THE TWISTED ENDOSCOPY OF $GL(4)$ AND $GL(5)$: TRANSFER OF SHALIKA GERMS

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Abstract. The transfer of germs of orbital integrals is carried out for the twisted endoscopy of $GL(4)$ and $GL(5)$, for p-adic fields of odd residual characteristic. We also transfer the κ-orbital integrals of Sp(4) to its endoscopic groups, again in odd residual characteristic.

Introduction. Kottwitz and Shelstad have general conjectures relating twisted orbital integrals on a reductive p-adic group to stable orbital integrals on endoscopic groups [KS]. The theory of descent, currently known only in the special case of standard endoscopy, reduces the transfer of orbital integrals to a local statement about the matching of Shalika germs at the identity, for the original reductive group together with the additional reductive groups obtained by descent [LS2].

This paper proves the needed statements about the matching of Shalika germs at the identity for the groups $GL(4)$ and $GL(5)$. Thus, the results of this paper, coupled with an expected theory of descent, would imply the transfer of twisted orbital integrals on $GL(4)$ and $GL(5)$ to stable orbital integrals on endoscopic groups.

These calculations give a first example of transfer in a *nonelementary* setting. The Shalika germs of the twisted orbital integrals on $GL(4)$ and $GL(5)$ are expressed by the number of points on a family of elliptic curves over finite fields. The stable orbital integrals on the endoscopic groups SO(5) and Sp(4) have a similar description. The transfer of orbital integrals to the endoscopic groups is established by producing isogenies between the families of elliptic curves. The presence of elliptic curves in a similar context was first noticed by Kazhdan, Lusztig, and Bernstein [KLB]. For further results along these lines, see [H3].

These calculations give complete formulas for the Shalika germs of Sp(4). By results of Kazhdan and Harish-Chandra, the Fourier transforms of subregular nilpotent orbits on the elliptic set coincide, up to a change of basis, with cuspidal combinations of subregular Shalika germs on the group Sp(4) (see [Ka],[HC]). Thus the Fourier transforms of certain subregular nilpotent orbits on elliptic elements of Sp(4), being expressed in terms of points on families of elliptic curves over finite fields, are not elementary.

The transfer of orbital integrals and the related fundamental lemmas are essential steps in the development of a stable trace formula. This paper brings us one step closer to obtaining a stable trace formula for Sp(4).

We also give the first verification of an unpublished conjecture of Assem and Kottwitz. Their conjecture states that the stable Shalika germ associated with a unipotent conjugacy class that is not special is identically zero. (For the definition of *special* unipotent classes, see [C].) We verify this conjecture for the *two-regular* unipotent conjugacy class in Sp(4).

Date: September 1993.
Duke Mathematical Journal 76.2 (1994): 595-632.

This verification is part of the proof of the transfer of twisted orbital integrals on $GL(4)$ and $GL(5)$.

The final section gives a proof of the transfer of orbital integrals on Sp(4) to its endoscopic groups. The paper [H1] carried this out for the group $GSp(4)$, but the analysis of the two-regular conjugacy class of Sp(4) was left incomplete. The formula for the germs of the two-regular conjugacy class, obtained as part of our work on the twisted theory, provides an essential ingredient in the transfer.

I would like to thank L. Clozel for his hospitality at the Université de Paris-Sud, and the C.N.R.S. for its generous support.

1. Twisted Endoscopy of the General Linear Group

Let θ be the automorphism of $GL(n)$ defined by $\theta(x) = J \cdot {}^t x^{-1} \cdot J^{-1}$, where J is the matrix with alternating signs along the skew diagonal and zeros elsewhere:

$$J = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \cdots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

The connected components of the fixed point sets of θ are the quasisplit groups $GL(2n)^{\theta\circ} = \mathrm{Sp}(2n)$, and $GL(2n+1)^{\theta\circ} = \mathrm{SO}(2n+1)$. One of the endoscopic groups for $(GL(2n), \theta)$ is $H = \mathrm{SO}(2n+1)$, and one of the endoscopic groups for $(GL(2n+1), \theta)$ is $H = \mathrm{Sp}(2n)$. By a result of Arthur [A,Lemma 2.1] generalizing Harish-Chandra, the stable twisted orbital integral near the unit element in $GL(2n)$ descends to a stable orbital integral on the fixed point set $\mathrm{Sp}(2n)$. Similarly, twisted orbital integrals on $GL(2n+1)$ descend to $\mathrm{SO}(2n+1)$.

The transfer factor of Kottwitz and Shelstad for the endoscopic groups just given is a constant that we can take to be 1 (see [KS]). (This is a general property of the "largest" twisted endoscopic group.) We are thus led to compare stable orbital integrals at the identity on $\mathrm{Sp}(2n)$ and $\mathrm{SO}(2n+1)$. Near the identity element, we may pass to the Lie algebra. We have an identification of split Cartan subalgebras in the Lie algebras:

$$\mathrm{diag}(t_1, \ldots, t_n, -t_n, \ldots, -t_1) \subset \mathfrak{sp}(2n)$$

may be identified with

$$\mathrm{diag}(t_1, \ldots, t_n, 0, -t_n, \ldots, -t_1) \subset \mathfrak{so}(2n+1).$$

We may also identify Weyl groups on these two groups, and this allows us to identify stable conjugacy classes of regular semisimple elements in $\mathfrak{sp}(2n)$ and $\mathfrak{so}(2n+1)$. If we express each stable Shalika germ on $\mathfrak{sp}(2n)$ as a linear combination of stable Shalika germs on $\mathfrak{so}(2n+1)$, and conversely, each stable Shalika germ on $\mathfrak{so}(2n+1)$ as linear combinations of stable Shalika germs on $\mathfrak{sp}(2n)$, then the transfer of twisted germs from $(GL(n), \theta)$ to the given endoscopic groups holds. Thus the twisted transfer manifests itself as a duality between groups of type B_n and groups of type C_n. This duality exchanges short and long roots and is nontrivial even in the rank two case we study.

It is easy to carry out this comparison of germs for two of the unipotent stable conjugacy classes of unipotent elements. With appropriate normalizations of measures the stable germ associated with the stable regular unipotent conjugacy class is identically 1 (see Shelstad [Sh]). Thus the stable germs associated with the regular unipotent class are equal for this normalization of measures. For a quite different normalization, the stable germ associated with the trivial unipotent class is zero for nonelliptic Cartan subgroups and constant on the elliptic Cartan subgroups [R]. Elliptic Cartan subgroups correspond in the two groups, so the matching holds in this case as well. There is a discriminant factor that comes into the different normalizations. Near the identity, the discriminant factors are equal, up to a constant, for $\mathfrak{sp}(2n)$ and $\mathfrak{so}(2n+1)$, and so these factors do not affect the conclusion.

The main result of this paper is the following theorem. It expresses a duality on groups of type B_2 that exchanges the short and long roots. As we have remarked, an immediate corollary of this result is the transfer of twisted stable Shalika germs from $GL(4)$ to SO(5) and from $GL(5)$ to Sp(4). In the following theorem we identify regular semisimple conjugacy classes in the Lie algebras $\mathfrak{so}(5)$ and $\mathfrak{sp}(4)$ as above.

Theorem 1.1. *Let F be a p-adic field of characteristic zero whose residual characteristic is odd. Each stable Shalika germ on $\mathfrak{so}(5)$ is a linear combination of Shalika germs on $\mathfrak{sp}(4)$, and conversely each stable Shalika germ on $\mathfrak{sp}(4)$ is a linear combination of Shalika germs on $\mathfrak{so}(5)$.*

In this rank two situation, we must compare stable orbital integrals on $\mathfrak{sp}(4)$ and $\mathfrak{so}(5)$. These Lie algebras are isomorphic. Thus, at the level of Lie algebras, we compare orbital integrals of parameters $(t_1, t_2, 0, -t_2, -t_1)$ and $(t_1+t_2, t_1-t_2, 0, t_2-t_1, -t_1-t_2)/2$. We consider these vectors as representing points in a nonsplit Cartan subalgebra of $\mathfrak{so}(5)$ defined over F, which has been diagonalized over the algebraic closure of F.

We say that a unipotent conjugacy class is k-regular if $2k = \dim C_G(u) - \mathrm{rank}(G)$, for an element u in the conjugacy class. It is known that k is a nonnegative integer. For the group SO(5) there are four unipotent conjugacy classes over the algebraic closure of F. These classes are regular (0-regular), subregular (1-regular), two-regular, and four-regular (the trivial unipotent element). We have already treated the regular and four-regular classes by a general argument. Section 3 will discuss the two-regular unipotent class. The stable germ of the subregular unipotent class was determined in [H2]. Let us recall the formula for $G = \mathrm{SO}(2n+1)$. Over the field F, the subregular unipotent classes are parametrized by $F^\times/F^{\times 2}$. Consider parameters $t_1, \ldots, t_n$ in the algebraic closure of F, such that the Weyl orbit of $(t_1, t_2, \ldots, -t_2, -t_1)$ is defined over F.

Theorem 1.2. *The stable Shalika germ corresponding to the coset $F^{\times 2}$ is the principal-value integral*

$$\int_F \log|(1-t_1^2x^2)(1-t_2^2x^2)\cdots(1-t_n^2x^2)| \left|\frac{dx}{x^2}\right|.$$

The stable Shalika germ corresponding to a nontrivial coset $aF^{\times 2}$ is

$$\int_{\mathcal{I}m} \eta((1-t_1^2u^2)(1-t_2^2u^2)\cdots(1-t_n^2u^2)) \left|\frac{du}{u^2}\right|.$$

Here η is the nontrivial quadratic character of $F^\times$ that is trivial on $x^2 - ay^2$, for $x, y \in F$, and $\mathcal{I}m = F\sqrt{a}$ is the corresponding imaginary axis.

Proof. This is a reformulation of [H2]. For the nontrivial cosets it is an easy reformulation. For the trivial class, the reformulation requires more work. Theorem 1.2 holds even if the residual characteristic is even. For further details about the normalization of the logarithm and absolute values see Section 2.

The stable subregular germ is an integral over the F-points of a connected surface S defined over F. The irreducible components of S are indexed by the irreducible components of the variety $\mathcal{B}_u$ of Borel subgroups containing a unipotent element u, the element u lying in the given conjugacy class. For SO$(2n+1)$, the irreducible components may be denoted $S_1, \ldots, S_n$ and $S'_1, \ldots, S'_{n-1}$. The components S_i and S_{i+1} intersect along a projective line, as do the components S'_i and S'_{i+1} and the components S'_{n-1} and S_n. There are no other intersections. Label the simple roots $\alpha_1, \ldots, \alpha_n$, according to adjacency in the Dynkin diagram with α_n a short root. We also write these roots as $\alpha_1 = t_1 - t_2$, $\alpha_2 = t_2 - t_3$, …, $\alpha_n = t_n$. Then the components S_i, for $i \le n$, and S'_i, for $i < n$, are indexed by the projective lines in $\mathcal{B}_u$ of type α_i.

Each irreducible component is a rational surface. We give a schematic representation of the surface in Diagram 1. Each square represents an irreducible component, and the intersections are represented by a common edge. In [H2], we developed two coordinate systems (w, ξ) for each irreducible component. Diagram 1 shows how to give distinguishing subscripts to the coordinates (w, ξ) on each irreducible component. For example, on the irreducible component associated with the simple root α_2, the coordinate system (w_{-2}, ξ_{-2}) is the system for which $\xi_{-2} = 0$ defines the intersection with the irreducible component associated with the simple root α_1. For the index 1, we have the coordinates $(z_{-1}, \xi_{-1}) = (z, \xi)$ introduced in [H2,VI,Prop. 3.2].

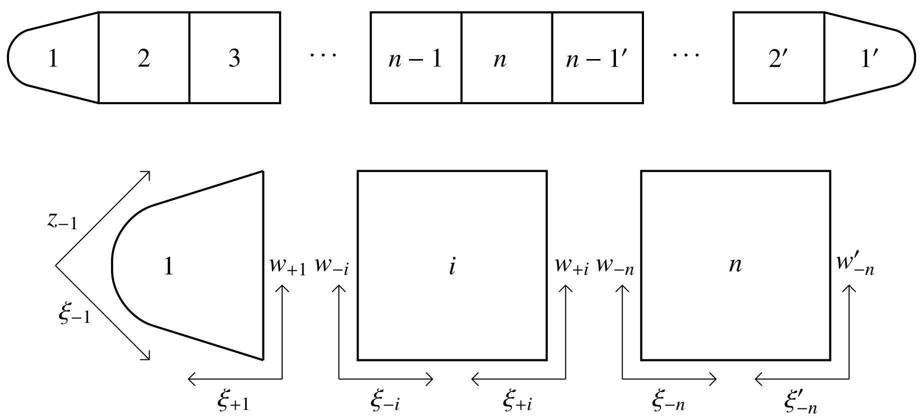

Figure 1. Local Coordinates on S

We will treat the irreducible components $S_1, \ldots, S_n$. The remaining components are treated in an identical manner, by adding primes to all of the coordinates. We set

$$x_{-i} = \frac{w_{-i}}{(1 - t_i w_{-i})}, \qquad y_{-i} = \frac{\xi_{-i}}{(1 + t_1 x_{-i}) \cdots (1 + t_i x_{-i})}, \quad \text{for } 1 < i \le n,$$
$$x_{+i} = \frac{w_{+i}}{(1 - t_{i+1} w_{+i})}, \quad y_{+i} = \xi_{+i}(1 + t_1 x_{+i}) \cdots (1 + t_i x_{+i}), \quad \text{for } 1 \le i < n.$$

Transporting the action of the Weyl group (described in [H2,VI,Theorem 1.6]) to this new set of coordinates, we find that the coordinates $x_{\pm i}, y_{\pm i}$ are all Weyl-group invariant. (To verify this, one must know explicit values for the constants $e(\alpha, \beta)$: $e(\alpha_i, \alpha_{i+1}) = 1$, $e(\alpha_i, \alpha_{i-1}) = -1$, for $i < n$, and $e(\alpha_n, \alpha_{n-1}) = -2$.) We may also take $\zeta = 1$, since SO$(2n+1)$ is quasisplit. The transition relations of [H2,V,Lemma 5.3,Lemma 6.1] imply that on the overlap of two coordinate charts on S_i, we have $x_{-i} = x_{+i}$ and $y_{-i} y_{+i} = 1$, for $i < n$. On S_n, we have $x_{-n} = -x'_{-n}$ and $y_{-n} y'_{-n} = 1/P(x_{-n})$, where $P(x_{-n}) := (1 - t_1^2 x_{-n}^2) \cdots (1 - t_n^2 x_{-n}^2)$.

The Galois group Gal $(\bar{F}/F)$ permutes the irreducible components of $\mathcal{B}_u$ and S in a compatible way. For SO$(2n+1)$ and the unipotent class associated with $a\,F^{\times 2}$, each irreducible component is defined over $E = F(\sqrt{a})$.

Now consider a nontrivial coset $a\,F^{\times 2}$. In this case, only one irreducible component S_n of S contains rational points. The action of the nontrivial element σ_0 in the Galois group Gal (E/F) on the coordinates is $\sigma_0(x_{-n}) = -x_{-n}$ and $\sigma_0(y_{-n})y_{-n} = 1/P(x_{-n})$. (See [H2,VI,Theorem 1.6]. There is a sign error in part (d) of that theorem: the right-hand side of the formula for $\sigma_0(w)$ should be negated.) The coordinates are Weyl-group invariant, so that these relations hold for all Cartan subgroups. The Shalika germ is then the integral over the surface S_n with respect to the measure $|x_{-n}^{-2}y_{-n}^{-1}dx_{-n}dy_{-n}|$. Since $\sigma_0(y_{-n})y_{-n} = 1/P(x_{-n})$, we see that $P(x_{-n})$ is a norm from E, so that

$$1 = \frac{1}{2}(1 + \eta(P(x_{-n}))).$$

We may also replace y_{-n} by a multiple of y_{-n} so that its norm is 1, independent of x_{-n}. Thus the germ is

$$\int_{S_n} \left|\frac{dy_{-n}\,dx_{-n}}{y_{-n}\;x_{-n}^2}\right| = \frac{1}{2}\int\left|\frac{dy_{-n}}{y_{-n}}\right|\int_{\mathbb{P}^1}\left|\frac{dx_{-n}}{x_{-n}^2}\right| + \frac{1}{2}\int\left|\frac{dy_{-n}}{y_{-n}}\right|\int_{\mathbb{P}^1}\eta(P(x_{-n}))\left|\frac{dx_{-n}}{x_{-n}^2}\right|.$$

By Lemma 2.2, the integral $\int|dx_{-n}/x_{-n}^2|$ is zero, so that the first of the two terms is zero. The normalizing factor $\int|dy_{-n}/y_{-n}|/2$ is independent of x_{-n} and t_i. Ignoring this constant, we obtain the formula of the theorem.

Now turn to the trivial coset $F^{\times 2}$. Each irreducible component is defined over F. We must compute the volume of the surface with respect to the given coordinates and the measure attached to the differential form

$$\frac{dx_{-i}\wedge dy_{-i}}{x_{-i}^2\;y_{-i}}.$$

The coordinates $x_{\pm i}$, $y_{\pm i}$ are defined over F for every Cartan subgroup, because the coordinates are Weyl-group invariant. The form has a simple pole along each intersection of adjacent irreducible components. One way to compute the volume is to pick coordinates in such a way that the only contributions to volume come from the poles. With the coordinates given above, the only contribution to volume will come from the irreducible component S_n.

We begin this calculation by showing that the contribution of each simple pole $\xi_{-i} = 0$, for $i > 1$, is zero. We follow the procedure and notation explained in [H2,p83] (compare [La,p469] and [H1,p238]). By [H2,V,6.1.d], on the irreducible component S_i, the coordinate y_{-i} is the restriction of

$$\frac{z(W_+,\alpha_i)}{(t_i - t_{i+1})w_{-i}e(\alpha_i,\alpha_{i-1})(1+t_1x_{-i})\cdots(1+t_ix_{-i})},$$

and $y_{+(i-1)}$ is the restriction of

$$\frac{z(W_+,\alpha_{i-1})(1+t_1x_{+(i-1)})\cdots(1+t_{i-1}x_{+(i-1)})}{(t_{i-1}-t_i)w_{+(i-1)}e(\alpha_{i-1},\alpha_i)}.$$

Along the pole, $w_{-i} = w_{+(i-1)}$, $x_{-i} = x_{+(i-1)}$, and the product $y_{-i}y_{-i}$ is the restriction of

$$\frac{z(W_+,\alpha_i)z(W_+,\alpha_{i-1})}{(t_i-t_{i+1})(t_{i-1}-t_i)w^2e(\alpha_i,\alpha_{i-1})e(\alpha_{i-1},\alpha_i)(1+t_ix_{-i})}. \tag{1.3}$$

For the definitions of λ, $z(W_+, \alpha)$, and so forth we refer the reader to [H2]. The quotient of λ by Expression 1.3 simplifies to

$$x(\gamma)w_{-i}(1 + t_i x_{-i}) = x(\gamma)x_{-i}.$$

We must then integrate the logarithm of the absolute value of this quotient over the pole $\xi_{-i} = 0$ of S_i using the differential form dx_{-i}/x_{-i}^2. Both $x(\gamma)$ and x_{-i} are defined over F. It then follows from the vanishing of the integrals

$$\int_F \log|x| \left|\frac{dx}{x^2}\right|, \qquad \int_F \left|\frac{dx}{x^2}\right|$$

that the simple poles contribute nothing to the principal-value integral.

Next we analyze the contributions from the remainder of the irreducible components S_i, $i \neq 1, n$. The contribution is obtained by truncating near the simple poles: $|y_{-i}| \geq \epsilon_1$, $|y_{+i}| \geq \epsilon_2$. As we have $y_{+i} = 1/y_{-i}$, we find that the contribution is (by Lemma 2.2)

$$\int_{\epsilon_2^{-1} \geq |y_{-i}| \geq \epsilon_1} \left|\frac{dy_{-i}}{y_{-i}}\right| \int_F \left|\frac{dx_{-i}}{x_{-i}^2}\right| = \int_{\epsilon_2^{-1} \geq |y_{-i}| \geq \epsilon_1} \left|\frac{dy_{-i}}{y_{-i}}\right| \cdot 0 = 0.$$

The irreducible component S_1 requires special treatment. On S_1 the coordinates $(z_{-1}, \xi_{-1}) = (z, \xi)$ are essentially the F-variables $x_0 = x_{+1}/y_{+1}$ and $y_0 := 1/y_{+1}$. The form is $x_0^{-2} dx_0 \wedge dy_0$. The component S_1 makes no contribution because the truncated integral $\int_{|y_0|<C} |x_0^{-2} dx_0\, dy_0|$ is zero. (Compare [H1,page237] where this argument is made in the rank two situation.)

The final contribution comes from the truncated integral on S_n. Here we truncate by $|y_{-n}| \geq \epsilon_1$ and $|y'_{-n}| \geq \epsilon_2$. Since $y_{-n}y'_{-n} = 1/P(x_{-n})$, we have a contribution

$$\int_{(\epsilon_2|P(x_{-n})|)^{-1} \geq |y_{-n}| \geq \epsilon_1} \left|\frac{dy_{-n}}{y_{-n}}\right| \int_F \left|\frac{dx_{-n}}{x_{-n}^2}\right|,$$

and this is a constant times

$$\int_F \log|P(x_{-n})| \left|\frac{dx_{-n}}{x_{-n}^2}\right|.$$

This completes the proof of Theorem 1.2. □

2. The Basic Identity for the Subregular Germs

Notation. Throughout the remainder of this paper, let F be a p-adic field of characteristic zero, of odd residual characteristic with fixed algebraic closure $\bar{F}$. Let O_F denote the ring of integers in F, and let π denote a uniformizing element in F. We often work with a fixed finite Galois extension K/F, with intermediate fields E, E_1, and so forth. Let $|\cdot|$ denote the normalized absolute value on F, extended to the field K. Define val(x), for $x \in K$, by $|x| = q^{-\mathrm{val}(x)}$, where q is the cardinality of the residue field of F. Thus $\mathrm{val}(x)e(K/F) \in \mathbb{Z}$, for $x \in K$, where $e(K/F)$ is the ramification index of the field extension K/F. We define $\log|x|$ to be $-\mathrm{val}(x)$.

We introduce further notation for this particular section. We enumerate the nontrivial quadratic extensions E_1, E_2, E_3 of F, selecting E_1 to be the unramified extension, and E_2, E_3 to be ramified. We select uniformizing elements $\pi_2 \in E_2$ and $\pi_3 \in E_3$. We may assume that $\pi_2^2 = \pi$, a fixed uniformizing element of F, and that $\pi_3^2 = \epsilon\pi$, where ϵ is a unit in F. Then E_1 is obtained by adjoining to F a square root of ϵ. The Galois groups Gal(E_i/F) of these quadratic extensions have nontrivial elements that we denote σ_i, for $i = 1, 2, 3$. The quadratic extensions have *imaginary* axes $\mathcal{I}m_i = \{u \in E_i | \sigma_i(u) = -u\}$, for $i = 1, 2, 3$. We let η_i be the quadratic character of $F^\times$ associated with E_i, for $i = 1, 2, 3$. We have $\pi_i \in \mathcal{I}m_i$, for $i = 2, 3$.

For $a, b \in F$, we let $P(x) = P(x, a, b)$ denote the polynomial $P(x) = 1 - ax^2 + bx^4$. We factor the polynomial as $P(x) = (1 - \alpha x^2)(1 - \beta x^2)$, where $\alpha = \alpha(a, b)$, $\beta = \beta(a, b)$ lie in some quadratic extension of F. We define a dual polynomial

$$\begin{aligned} P'(x, a, b) &= P(x, 2a, a^2 - 4b) = 1 - 2ax^2 + (a^2 - 4b)x^4 = (1 - a'x^2 + b'x^4) \\ &= (1 - (\alpha^{1/2} - \beta^{1/2})^2 x^2)(1 - (\alpha^{1/2} + \beta^{1/2})^2 x^2) \\ &= (1 - \alpha' x^2)(1 - \beta' x^2), \end{aligned}$$

where $\alpha' = \alpha + \beta - 2\sqrt{b} = a - 2\sqrt{b}$, and $\beta' = \alpha + \beta + 2\sqrt{b} = a + 2\sqrt{b}$. Also, $a' = 2a$, $b' = a^2 - 4b = \alpha'\beta' = (\alpha - \beta)^2$. The constants $a, b, a', b', \alpha, \beta, \alpha', \beta'$ will satisfy these fixed relations throughout the section.

We consider the one-form dx on the affine line and the associated measure $|dx|$ on F. We set

$$\Gamma_0(a, b) = \int_F \log |P(x, a, b)| \left| \frac{dx}{x^2} \right|.$$

This and all the integrals that follow are to be interpreted as a principal-value integrals. Extending the absolute value $| \cdot |$ and the associated measure to field extensions E_i/F, we also set

$$\Gamma_i(a, b) = \int_{\mathcal{I}m_i} \eta_i(P(u, a, b)) \left| \frac{du}{u^2} \right|.$$

We normalize these integrals by setting $\bar{\Gamma}_i(a, b) = \Gamma_i(a, b)/\Gamma_i(\epsilon, 0)$, for $i = 0, 1, 2, 3$. We have seen that these integrals are the stable subregular Shalika germs of SO(5). The integrals Γ_2 and Γ_3 are *elliptic* in the sense that for certain choices of parameters $a, b \in F$ the values of the integrals are expressed by the number of points on elliptic curves over finite fields.

Set $s_q = 1$, if -1 is a square in F, and set $s_q = -1$ otherwise.

We are now ready to state the main technical result of this paper. By Theorem 1.2, this result implies the matching of stable subregular Shalika germs. It expresses the nontrivial part of the transfer of Shalika germs from the twisted reductive groups $GL(4)$ and $GL(5)$ to endoscopic groups SO(5) and Sp(4). The geometrical content of this proposition is the existence of particular isogenies of elliptic curves.

Proposition 2.1. *For all $a, b \in F$,*

$$\begin{pmatrix} \bar{\Gamma}_0(2a, a^2 - 4b) \\ \bar{\Gamma}_1(2a, a^2 - 4b) \\ \bar{\Gamma}_2(2a, a^2 - 4b) \\ \bar{\Gamma}_3(2a, a^2 - 4b) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & s_q & -s_q \\ 1 & -1 & -s_q & s_q \end{pmatrix} \begin{pmatrix} \bar{\Gamma}_0(a, b) \\ \bar{\Gamma}_1(a, b) \\ \bar{\Gamma}_2(a, b) \\ \bar{\Gamma}_3(a, b) \end{pmatrix}.$$

This proposition expresses a duality in the sense that the statement is equivalent to the identity

$$\begin{pmatrix} \bar{\Gamma}_0(a, b) \\ \bar{\Gamma}_1(a, b) \\ \bar{\Gamma}_2(a, b) \\ \bar{\Gamma}_3(a, b) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & s_q & -s_q \\ 1 & -1 & -s_q & s_q \end{pmatrix} \begin{pmatrix} \bar{\Gamma}_0(2a, a^2 - 4b) \\ \bar{\Gamma}_1(2a, a^2 - 4b) \\ \bar{\Gamma}_2(2a, a^2 - 4b) \\ \bar{\Gamma}_3(2a, a^2 - 4b) \end{pmatrix},$$

obtained by taking the inverse of the 4×4 matrix. We could also substitute a' for a and b' for b and note that $\bar{\Gamma}_i(2a', a'^2 - 4b') = \bar{\Gamma}_i(4a, 16b) = \bar{\Gamma}_i(a, b)$ because $|2| = 1$. Note that the 4×4 matrix is the matrix $\eta_i(x_j)$ obtained by pairing the trivial character η_0 and the nontrivial characters η_1, η_2, η_3 with the representatives $x_j = 1, \epsilon, \pi, \epsilon\pi$ of the cosets of $F^\times / F^{\times 2}$.

The proposition is proved by calculating the integrals on both sides for all possible choices of parameters $a, b \in F$, and comparing the two sides. We will see that the integrals $\bar{\Gamma}_i(a, b)$ are linearly independent over $\mathbb{C}$, so that the 4×4 matrix in the proposition is unique.

The most difficult of these integrals to compute are $\Gamma_i(a, b)$, for $i = 2, 3$. In the typical case, $|\alpha| = |\beta| = |\alpha-\beta|$. Let us briefly describe how it is calculated, say for $i = 2$. There will be a large number of integrals such as this to compute, and we will leave the routine details of the calculations to the reader. Under the substitution $u = \pi_2 x$, the integral becomes

$$|\pi_2|^{-1} \int_F \eta_2(P(x, \bar{a}, \bar{b})) \left|\frac{dx}{x^2}\right|,$$

where $\bar{a} = \pi_2^2 a$, $\bar{b} = \pi_2^4 b$, $\bar{\alpha} = \pi_2^2\alpha$, and $\bar{\beta} = \pi_2^2\beta$.

On the interval $|\bar{\alpha}x^2| > 1$, we have $\eta_2(P(x, \bar{a}, \bar{b})) = \eta_2(\bar{b})$, a constant, and the integral becomes a simple geometric series. On the interval $|\bar{\alpha}x^2| < 1$, we have $\eta_2(P(x, \bar{a}, \bar{b})) = 1$, and again we have a geometric series that is easily summed. On the set $\{x : |\bar{\alpha}x^2| = 1;\ |P(x, \bar{a}, \bar{b})| = 1\}$, we see that $\eta_2(P(x, \bar{a}, \bar{b}))$ depends only on whether the reduction to the residue field of $P(x, \bar{a}, \bar{b})$ is a square, so the integral will be expressed in terms of the number of affine points (with $\bar{y} \neq 0$) on the curve over the finite field $\bar{y}^2 = \bar{P}(\bar{x})$, obtained by reducing $P(x, \bar{a}, \bar{b})$. This explains the appearance of elliptic curves.

Finally we must consider elements x such that $|\bar{\alpha}x| = 1$ and $|P(x, \bar{a}, \bar{b})| < 1$. Call this the set of *approximate roots*. We have a local coordinate $t = \bar{\alpha}(x - x_0)$, with $|t| < 1$, near a fixed approximate root x_0. The integral is a constant times

$$\int_{|t|<1} \eta_2(t)|dt|,$$

and this is zero because η_2 is ramified. So the set of approximate roots contributes nothing to the integral. When we use the unramified character η_1, however, there is in general a nonzero contribution from the set of approximate roots. See Lemma 2.6.

The following vanishing result, which we have already invoked more than once, will be used frequently in this paper. A proof is found in [LS1].

LEMMA 2.2. *Let $\theta : F^\times \to \mathbb{C}^\times$ be a nontrivial quasicharacter of $F^\times$. Then*

$$\int_{\mathbb{P}^1} \theta(x) \left|\frac{dx}{x}\right| = 0.$$

It is often easier to compute the integral of $\eta_i(P) - 1$ than the integral of $\eta_i(P)$. The two integrands give the same result because of the lemma. If, for example, $\eta_2(\bar{b}) = 1$, then $\eta_2(P(x, \bar{a}, \bar{b})) - 1 = 0$, except when $|\bar{\alpha}x^2| = 1$, so that the only contributions to the integral come from the curve over the finite field.

Rather than show the calculation of all of the integrals, we will show how to express them in terms of integrals whose values have been tabulated. The following integrals are tabulated in the appendix.

$$L(a) := \int_F \log|1 - ax^2| \left|\frac{dx}{x^2}\right|, \quad \text{for } a \in F.$$

$$H_i(a) := \int_F \eta_i(1 - ax^2) \left|\frac{dx}{x^2}\right|, \quad \text{for } a \in F \text{ and } i = 1, 2, 3.$$

In terms of these integrals, the normalizing constants entering into the definitions of the functions $\bar{\Gamma}_i(a, b)$ are $\Gamma_0(\epsilon, 0) = L(\epsilon)$, $\Gamma_1(\epsilon, 0) = H_1(1)$, $\Gamma_2(\epsilon, 0) = H_2(\epsilon\pi_2^2)/|\pi_2| = \sqrt{q}H_2(\epsilon\pi)$, $\Gamma_3(\epsilon, 0) = \sqrt{q}H_3(\pi)$. Thus, by the tables in the appendix, the normalized integrals $\bar{\Gamma}_i(a, b)$ stand in explicit relation to $\Gamma_i(a, b)$.

Lemma 2.3. *It is enough to establish the Proposition 2.1 in the following four cases.*

(1) $|\alpha| > |\beta|$, *and* $\alpha, \beta \in F$.
(2) $|\alpha| = |\beta| = |\alpha - \beta| = |\alpha'| = |\beta'| = |\alpha' - \beta'|$, *and* $\mathrm{val}(\alpha) \in \mathbb{Z} + \frac{1}{2}$.
(3) $|\alpha| = |\beta| = |\alpha - \beta| = |\alpha'| = |\beta'| = |\alpha' - \beta'|$, *and* $\mathrm{val}(\alpha) \in 2\mathbb{Z}$.
(4) $|\alpha| = |\beta| = |\alpha - \beta| = |\alpha'| = |\beta'| = |\alpha' - \beta'|$, *and* $\mathrm{val}(\alpha) \in 2\mathbb{Z} + 1$.

Proof. First we dispense with the degenerate case when $\beta = 0$. It is easily deduced from the tables of the appendix that $\bar{\Gamma}_i(a, 0)$ is independent of i. The proposition reduces to the vanishing integral of Lemma 2.2 and the claim

$$\bar{\Gamma}_0(2a, a^2) = 2\bar{\Gamma}_0(a, 0).$$

This is an immediate consequence of the identity $\log|1 - 2ax^2 + a^2x^4| = 2\log|1 - ax^2|$. Thus we assume that $\beta \neq 0$, by symmetry that $\alpha \neq 0$, $b \neq 0$, and, by the duality following the statement of Proposition 2.1, that $\alpha' \neq 0, \beta' \neq 0$, and $b' = (\alpha - \beta)^2 \neq 0$.

Since α and β are the roots of a quadratic polynomial over F, if the roots are not in F, then their valuations are equal. If $|\alpha| \neq |\beta|$, then by symmetry we may take $|\alpha| > |\beta|$.

Now assume that $|\alpha| = |\beta|$. If $|\alpha'| \neq |\beta'|$, then by symmetry we can assume that $|\alpha'| > |\beta'|$, and then by the duality explained after the statement of the proposition, we may conclude the proposition from Case 1. So we may assume that ${\alpha'}^{1/2} = (\alpha^{1/2} - \beta^{1/2})$ and ${\beta'}^{1/2} = (\alpha^{1/2} + \beta^{1/2})$ have equal valuations. Their valuation must equal that of either their sum or difference, so $|\alpha^{1/2} - \beta^{1/2}| = |\alpha^{1/2} + \beta^{1/2}| = |\alpha^{1/2}| = |\beta^{1/2}|$. Thus, $|\alpha| = |\beta| = |\alpha - \beta| = |\alpha'| = |\beta'| = |\alpha' - \beta'|$. Now Cases 2, 3, and 4 give the three possibilities for the valuation of an element in a quadratic extension of F. □

Subregular Case 1. Unequal Valuation.

We now check Proposition 2.1 in Case 1 : $|\alpha| > |\beta|$, and $\alpha, \beta \in F$. Dropping subscripts, we let η be a quadratic character associated with a quadratic extension E, and let $\mathcal{I}m$ be the imaginary axis of E.

Lemma 2.4. *Suppose that* $|\alpha| > |\beta|$, *with* $\alpha, \beta \in F$. *Then*

$$\int_F \log|(1 - \alpha x^2)(1 - \beta x^2)| \left|\frac{dx}{x^2}\right| = L(\alpha) + L(\beta),$$

and

$$\int_{\mathcal{I}m} \eta((1 - \alpha u^2)(1 - \beta u^2)) \left|\frac{du}{u^2}\right| = \int_{\mathcal{I}m} \eta(1 - \alpha u^2) \left|\frac{du}{u^2}\right| + \eta(\alpha) \int_{\mathcal{I}m} \eta(1 - \beta u^2) \left|\frac{du}{u^2}\right|.$$

Proof. The first statement follows immediately from the identity $\log|xy| = \log|x| + \log|y|$, and the definition of the integral L. On the interval $|u|^2 < 1/|\beta|$, we have $\eta(P(u)) = \eta(1 - \alpha u^2)$, and $\eta(\alpha) = \eta(\alpha)\eta(1 - \beta u^2)$. And on the interval $|u|^2 \geq 1/|\beta|$, we have $\eta(P(u)) = \eta(\alpha)\eta(1 - \beta u^2)$, and $\eta(\alpha) = \eta(1 - \alpha u^2)$. Thus

$$\eta(P(u)) + \eta(\alpha) = \eta(1 - \alpha u^2) + \eta(\alpha)\eta(1 - \beta u^2),$$

for all $u \in \mathcal{I}m$. Now integrate over the imaginary axis, and use Lemma 2.2 to complete the proof: $\int_{\mathcal{I}m} \eta(\alpha)|du/u^2| = 0$. □

With a parametrization of the imaginary axis by $u = cx$, for some fixed $c \in \mathcal{I}m$ and $x \in F$, the integrals on the right-hand side in Lemma 2.4 equal those tabulated in the appendix. Thus, the integrals $\bar\Gamma_i(a,b)$ in Case 1 are known functions of q.

Now we turn to the dual direction. Recall that $P'(x) = (1 - 2ax^2 + (a^2 - 4b)x^4) = (1-\alpha' x^2)(1-\beta' x^2)$. The first case of Proposition 2.1 is a direct consequence of the following lemma and the previous one, because together they give explicit formulas for the functions $\bar\Gamma_i$. We leave the details of the verification to the reader.

Lemma 2.5. *Suppose that $|\alpha| > |\beta|$, and that $\alpha,\beta \in F$. Then*
(1) $\int_{\mathcal{I}m} \eta(P'(u))|du/u^2| = 0$, *if* $\alpha^{1/2} \notin \mathcal{I}m$.
(2) $\int_{\mathcal{I}m} \eta(P'(u))|du/u^2| = 2\int_{\mathcal{I}m} \eta(1-\beta u^2)|du/u^2|$, *if* $\alpha^{1/2} \in \mathcal{I}m$.
(3) $\int_F \log|P'(x)||dx/x^2| = 2\int_F \log|1-\alpha x^2||dx/x^2|$, *if* $\alpha \notin F^{\times 2}$.
(4) $\int_F \log|P'(x)||dx/x^2| = 2\int_F \log|1-\beta x^2||dx/x^2|$, *if* $\alpha \in F^{\times 2}$.

Proof. Since $\alpha' = (\alpha^{1/2} - \beta^{1/2})^2$, and $|\alpha| > |\beta|$, we have $|\alpha'| = |\alpha|$. Similarly, $|\beta'| = |\alpha|$. So $|\alpha'| = |\beta'|$. On the interval $|u|^2 < 1/|\beta'|$, we see that $\eta(P'(u)) = 1$, and on the interval $|u|^2 > 1/|\beta'|$, we see that $\eta(P'(u)) = \eta(b') = 1$, since $b' = (\alpha-\beta)^2 \in F^{\times 2}$.

On the set $\{u : |u|^2 = 1/|\beta'|, \text{ and } |P'(u)| = 1\}$, we see that $\eta(P'(u)) = \eta((1-\alpha u^2)^2) = 1$. So $\eta(P'(u)) = 1$, for all $u \in \mathcal{I}m$, except at points u in the imaginary axis such that $|1-\alpha u^2| < 1$. There are no such points if $\alpha^{1/2} \notin \mathcal{I}m$. In this case, the integral of Part 1 of the lemma is equal to $\int_{\mathcal{I}m} |du/u^2|$ and hence is zero by Lemma 2.2.

Assume that $\alpha^{1/2} \in \mathcal{I}m$. The integral of Part 2 the lemma is equal to

$$\int_{|1-\alpha u^2|<1} \eta(P'(u)) - 1 \left|\frac{du}{u^2}\right|.$$

Write $x \sim y$ if $xy^{-1} \in F^{\times 2}$. If $|x - y| < 1$ and $|x| \geq 1$, for $x, y \in F$, then $x \sim y$.

We have $P'(u) = ((1-\alpha^{1/2}u)^2 - \beta u^2)((1+\alpha^{1/2}u)^2 - \beta u^2)$. Thus if $|1-\alpha^{1/2}u| < 1$, then $P'(u) \sim ((1-\alpha^{1/2}u)^2 - \beta u^2)2^2 \sim (1-\beta v^2)$, where $v = u/(1-\alpha^{1/2}u)$. Similarly, if $|1+\alpha^{1/2}u| < 1$, then $P'(u) \sim (1-\beta v'^2)$, where $v' = u/(1+\alpha^{1/2}u)$. Making this change of coordinates we have

$$\int_{|1-\alpha u^2|<1} \eta(P'(u)) - 1 \left|\frac{du}{u^2}\right| = 2\int_{\mathcal{I}m} \eta(1-\beta v^2) - 1 \left|\frac{dv}{v^2}\right|.$$

We have used the fact that v and v' make equal contributions to the integral. We have also used the fact that $|1-\alpha^{1/2}u| < 1$ if and only if $|1+\alpha^{1/2}v|$, or equivalently $|\alpha^{1/2}v|$, is greater than 1. But if $|\alpha^{1/2}v| \leq 1$, then $1-\beta v^2 \sim 1$, so the integrand vanishes. This justifies the integration over the full imaginary axis. Part 2 now follows from Lemma 2.2.

Now turn to the log terms. We easily check that

$$\begin{aligned}\int_F |P'(x)|\left|\frac{dx}{x^2}\right| &= \int_{|\beta' x^2|>1} \log|b'x^4|\left|\frac{dx}{x^2}\right| + \int_{|\beta' x^2|=1,\ |P'(x)|<1} \log|P'(x)|\left|\frac{dx}{x^2}\right| \\ &= 2\int_{|\alpha x^2|>1} \log|\alpha x^2|\left|\frac{dx}{x^2}\right| + \int_{|\beta' x^2|=1,\ |P'(x)|<1} \log|P'(x)|\left|\frac{dx}{x^2}\right|.\end{aligned}$$

If α is not a square, the first term is $2\int_F \log|1-\alpha x^2||dx/x^2|$, and the second term is zero. This proves Part 3 of the lemma. Now assume that α is a square. Under the obvious symmetry $\alpha^{1/2} \leftrightarrow -\alpha^{1/2}$, we have

$$\int_{|\beta' x^2|=1,\ |P'(x)|<1} \log|P'(x)|\left|\frac{dx}{x^2}\right| = 2|\alpha|\int_{|1+\alpha^{1/2}x|<1} \log|P'(x)|\,|dx|.$$

If we set $v = x/(1+\alpha^{1/2}x)$, then whenever $|1+\alpha^{1/2}x| < 1$ we have

$$\log|P'(x)| = \log|2^2((1+\alpha^{1/2}x)^2 - \beta x^2)| = 2\log|1+\alpha^{1/2}x| + \log|1-\beta v^2|.$$

The integral of the second term is $2\int_F \log|1-\beta v^2|\,|dv/v^2|$. The integral of the first term becomes $-2\int_F \log|1-\epsilon\alpha t^2|\,|dt/t^2|$, under the substitution $1+\alpha^{1/2}x = 1/(t\alpha^{1/2})$. The result follows. □

Subregular Case 2. $\frac{1}{2}$-Integral Valuation.

Next we check the proposition in Case 2: $\mathrm{val}(\alpha) \in \mathbb{Z} + \frac{1}{2}$. The root α lies in a ramified quadratic extension of F. If $\alpha \in E_2 \setminus F$, then $b \in -\pi F^{\times 2}$, and if $\alpha \in E_3 \setminus F$, then $b \in -\pi\epsilon F^{\times 2}$, so the quadratic field containing α is determined by the class of b in $F^\times/F^{\times 2}$. Case 2 of Proposition 2.1 is a direct consequence of Part 3 of the following lemma.

Lemma 2.6. *(1) If $b \in -\pi\epsilon F^{\times 2}$, then*

$$\begin{aligned}
\Gamma_0(a,b) &= \frac{|b\pi^{-1}|^{1/4}(3q+1)}{q(q-1)}, \text{ if } \mathrm{val}(b) \equiv 1 \mod 4,\\
\Gamma_0(a,b) &= \frac{|b\pi^{-3}|^{1/4}(q+3)}{q(q-1)}, \text{ if } \mathrm{val}(b) \equiv 3 \mod 4,\\
\Gamma_1(a,b) &= \frac{-2|b\pi^{-j}|^{1/4}}{q}, \text{ if } \mathrm{val}(b) \equiv j \mod 4 \text{ and } j = 1,3,\\
\Gamma_2(a,b) &= \frac{-2|b\pi^{-1}|^{1/4}\sqrt{q}}{q}, \text{ if } \mathrm{val}(b) \equiv 1 \mod 4,\\
\Gamma_2(a,b) &= \frac{-2|b\pi^{-3}|^{1/4}\sqrt{q}}{q^2}, \text{ if } \mathrm{val}(b) \equiv 3 \mod 4,\\
\Gamma_3(a,b) &= 0.
\end{aligned}$$

(2) If $b \in -\pi F^{\times 2}$, then the same relations hold after interchanging the subscripts 2 *and* 3 *in Part 1.*

(3)

$$\begin{aligned}
&\Gamma_i(2a, a^2-4b) = \Gamma_i(a,b), \text{ for } i = 0,1,\\
&\Gamma_i(2a, a^2-4b) = \Gamma_i(a,b), \text{ for } i = 2,3, \text{ if } -1 \text{ is a square,}\\
&\Gamma_i(2a, a^2-4b) = \Gamma_j(a,b), \text{ for } (i,j) = (2,3),(3,2), \text{ if } -1 \text{ is not a square.}
\end{aligned}$$

Proof. The integrals of Part 1 are easily computed. Assume $\alpha \in E_3 \setminus F$. If $u \in \mathcal{I}m_3$, then $P(u)$ is the norm of $(1-\alpha u^2)$, so that $\eta_3(P(u)) = 1$, and $\Gamma_3(a,b) = 0$ by Lemma 2.2. The other integrals quickly reduce to geometric series and the integrals of the appendix. We omit the details. We show that Parts 1 and 2 imply Part 3. Note that $\Gamma_i(a,b)$ is independent of a and depends only on b modulo the multiplicative group of the squares of units, and that Γ_0 and Γ_1 depend only on the valuation of b. The elements b and b' have the same valuation, so $\Gamma_i(a,b) = \Gamma_i(a',b')$, for $i = 0,1$. Suppose that -1 is a square and that $b \in -\pi\epsilon F^{\times 2}$. Then $a^2 - 4b = b' = (\alpha-\beta)^2$ is the norm of $(\alpha - \sigma_3(\alpha))$. From this it follows that b'/b is the square of a unit, so $\Gamma_i(a,b) = \Gamma_i(a',b')$. On the other hand, if -1 is not a square, then b'/b

is a unit but not a square, so that the subscripts 2 and 3 must be interchanged in passing from $\Gamma_i(a', b')$ to $\Gamma_i(a, b)$. ◻

Subregular Case 3. Even Valuation.

Now we turn to the third case of the proof of the proposition, the case of parameters with even valuation. This is technically the most difficult. Let r_1 be the number of distinct roots in F of the polynomial $(1-\alpha x^2)(1-\beta x^2)$. Let r_ϵ be the number of distinct roots in F of the polynomial $P_\epsilon(x) := (1-\epsilon\alpha x^2)(1-\epsilon\beta x^2)$. Clearly $r_1, r_\epsilon \in \{0, 2, 4\}$ and $r_1 + r_\epsilon \leq 4$. The valuations of $b = \alpha\beta$ and $b' = \alpha'\beta'$ are multiples of 4, and $b' = (\alpha-\beta)^2 \in F$. So either $\alpha - \beta \in F$, or $\alpha - \beta \in \mathcal{I}m_1$.

Lemma 2.7. *In the context of Case 3, we have*

$$\Gamma_0(a,b) = \frac{(4-r_1)|b|^{1/4}}{q-1},$$
$$\Gamma_1(a,b) = \frac{-2r_\epsilon|b|^{1/4}}{q+1},$$
$$\Gamma_2(a,b) = \frac{-\sqrt{q}(1-\eta_2(b))|b|^{1/4}}{q},$$
$$\Gamma_3(a,b) = \Gamma_2(a,b).$$

Proof. These are routine calculations. The constants r_1 and r_ϵ enter into the result through the contribution from the approximate zeros of P and P_ϵ. ◻

We can deduce the entries of the following table from the lemma. The meaning of the various rows is described below. We set $A = |b|^{-1/4}\bar\Gamma_0(a,b)/2 = 1 - r_1/4$, $B = |b|^{-1/4}\bar\Gamma_1(a,b)/2 = r_\epsilon/4$, $C = |b|^{-1/4}(\bar\Gamma_2 + \bar\Gamma_3)/2 = (1-\eta_2(b))/2$. Set A', B', C' equal to $|b'|^{-1/4}\bar\Gamma_i(a', b')$, for $i = 0, 1, 2$. Then $A' = 2 - r'_1/2$, $B' = r'_\epsilon/2$, and $C' = (1-\eta_2(b'))/2$. Proposition 2.1, in this third case, then follows from the immediate verifications that $A' = A + B + C$, $B' = A + B - C$, $C' = A - B$.

	A	B	C	A'	B'	C'
(3*a*)	0	0	0	0	0	0
(3*b*)	1/2	1/2	1	2	0	0
(3*c*)	1	1	0	2	2	0
(3*d*)	1	0	0	1	1	1
(3*e*)	1	0	1	2	0	1

In the following explanations of the table we assume that $\alpha_0, \beta_0, u, v, w \in F^\times$. Row 3a is the subcase $\alpha, \beta \in F^{\times 2}$. Then $r_\epsilon = 0$, $r_1 = 4$, and $\eta_2(b) = 1$. Set $\alpha = \alpha_0^2$ and $\beta = \beta_0^2$. Then $\alpha' = (\alpha_0 - \beta_0)^2$ and $\beta' = (\alpha_0 + \beta_0)^2$. Thus, $r'_\epsilon = 0$, $r'_1 = 4$, and $\eta_2(b') = \eta_2(b) = 1$.

Row 3b is the subcase $\alpha = \epsilon\alpha_0^2$, and $\beta = \beta_0^2$. Then $\alpha' = (\beta_0 - \sqrt{\epsilon}\alpha_0)^2$, and $\beta' = (\beta_0 + \sqrt{\epsilon}\alpha_0)^2$. Thus $r_\epsilon = 2$, $r_1 = 2$, $r'_\epsilon = 0$, and $r'_1 = 0$. Furthermore, $\eta_2(b) = -1$, and $\eta_2(b') = 1$.

Row 3c is the subcase $\alpha = \epsilon\alpha_0^2$ and $\beta = \epsilon\beta_0^2$. Then $\alpha' = \epsilon(\alpha_0 - \beta_0)^2$, and $\beta' = \epsilon(\alpha_0 + \beta_0)^2$. Thus $r_\epsilon = 4$, $r_1 = 0$, $r'_\epsilon = 4$, $r'_1 = 0$, $\eta_2(b) = 1$, and $\eta_2(b') = 1$.

Row 3d is the subcase $\alpha = u - \sqrt{\epsilon}v$, $\beta = u + \sqrt{\epsilon}v$, $\alpha\beta = w^2$. Then $\alpha' = 2u - 2w$, $\beta' = 2u + 2w$, $w^2 = u^2 - \epsilon v^2$, and $b' = \alpha'\beta' = 4(u^2 - w^2) = 4\epsilon v^2$. Thus $\eta_2(b) = 1$, $\eta_2(b') = -1$, $r_1 = r_\epsilon = 0$, and $r'_1 = r'_\epsilon = 2$.

Row (3e) is the subcase $\alpha = u - \sqrt{\epsilon}v, \beta = u + \sqrt{\epsilon}v, \alpha\beta = \epsilon w^2$. Then $\alpha' = 2u - 2w\sqrt{\epsilon}$, $\beta' = 2u + 2w\sqrt{\epsilon}$, and $b' = \alpha'\beta' = 4(u^2 - w^2\epsilon) = 4\epsilon v^2$. Thus $r_1 = r_\epsilon = 0$, $r_1' = r_\epsilon' = 0$, and $\eta_2(b) = \eta_2(b') = -1$.

Subregular Case 4. Odd Valuation.

We are now ready to consider the proof of the final and most interesting case of the proposition. We assume that $|\alpha| = |\beta| = |\alpha - \beta| = |\alpha'| = |\beta'| = |\alpha' - \beta'|$. Let $\bar{\epsilon} \in k^\times \setminus k^{\times 2}$ be the reduction of ϵ to the residue field k. Let $\bar{\alpha} \in k^\times$ be the reduction of $\pi^{-\operatorname{val}(\alpha)}\alpha$ to its residue field. Similarly, for $x = a, b, a', b', \alpha, \beta, \alpha', \beta'$, we let $\bar{x}$ denote the reduction of $\pi^{-\operatorname{val}(x)}x$ to its residue field.

We let $C(\bar{a}, \bar{b})$ denote the number of k-points on the elliptic curve over k determined by the affine equation $\bar{\epsilon}y^2 = 1 - \bar{a}x^2 + \bar{b}x^4$, with $\bar{a}, \bar{b} \in k$.

Lemma 2.8. *In the context of Case 4,*

(1) $$\bar{\Gamma}_0(a,b) = \frac{(q+1)|b\pi^{-2}|^{1/4}}{q},$$

(2) $$\bar{\Gamma}_1(a,b) = 0,$$

(3) $$\bar{\Gamma}_2(a,b) = \frac{C(\bar{a},\bar{b})|b\pi^{-2}|^{1/4}}{2q},$$

(4) $$\bar{\Gamma}_3(a,b) = \frac{C(\bar{\epsilon}\bar{a},\bar{\epsilon}^2\bar{b})|b\pi^{-2}|^{1/4}}{2q},$$

(5) $$\bar{\Gamma}_0(a,b) = \bar{\Gamma}_2(a,b) + \bar{\Gamma}_3(a,b).$$

Proof. Again this is a routine calculation using the integration techniques described at the beginning of Section 2. □

Proof. We now verify the proposition in Case 4. The first two rows of the matrix in Proposition 2.1 are formal consequences of the lemma and the identity $|b'| = |b|$. Adding the last two rows of the matrix of the proposition, we find that we should have

$$\bar{\Gamma}_0(a,b) - \bar{\Gamma}_1(a,b) = \bar{\Gamma}_2(a',b') + \bar{\Gamma}_3(a',b').$$

This too is a formal consequence of the lemma. The proposition will then follow from the identity coming from the third row of the matrix, namely,

$$\bar{\Gamma}_0(a,b) - \bar{\Gamma}_1(a,b) + s_q\bar{\Gamma}_2(a,b) - s_q\bar{\Gamma}_3(a,b) = 2\bar{\Gamma}_2(a',b').$$

Or equivalently, by the lemma, we must show

$$\frac{(1+s_q)}{2}C(\bar{a},\bar{b}) + \frac{(1-s_q)}{2}C(\bar{\epsilon}\bar{a},\bar{\epsilon}^2\bar{b}) = C(\bar{a}',\bar{b}').$$

The left-hand side is always $C(-\bar{a},\bar{b})$. This is clear if -1 is a square because then $s_q = 1$ and $C(-\bar{a},\bar{b}) = C(\bar{a},\bar{b})$, under the substitution $x \mapsto x\sqrt{-1}$. If -1 is not a square, then $s_q = -1$, and so by taking $\bar{\epsilon} = -1$, we have $C(\bar{\epsilon}\bar{a},\bar{\epsilon}^2\bar{b}) = C(-\bar{a},\bar{b})$.

Thus we need to prove that there is the same number of points on the elliptic curve $\bar{\epsilon}y^2 = 1 + \bar{a}x^2 + \bar{b}x^4$ as on the curve $\bar{\epsilon}y^2 = 1 - \bar{a}'x^2 + \bar{b}'x^4$, or equivalently, that there is the same number of points on the elliptic curve $y_1^2 = 1 + \bar{a}x_1^2 + \bar{b}x_1^4$ as on the curve

$y_2^2 = 1 - \bar{a}' x_2^2 + \bar{b}' x_2^4$, where $\bar{a}' = 2\bar{a}$ and $\bar{b}' = \bar{a}^2 - 4\bar{b}$. This follows from the pair of dual isogenies of degree two between the curves:

$$\phi^* x_2 = x_1/y_1, \qquad \phi^* y_2 = (1 - \bar{b} x_1^4)/y_1^2,$$
$$\psi^* x_1 = 2x_2/y_2, \qquad \psi^* y_1 = (1 - \bar{b}' x_2^4)/y_2^2.$$

This completes the proof of the matching of the stable subregular Shalika germs. □

3. Two-Regular Shalika Germs

We now shift our attention from the subregular unipotent class to the two-regular unipotent class. In order to make use of the results of [H1], we now work with the group Sp(4) rather than SO(5). It makes no difference which we work with, if we bear in mind that the parameter $\mathrm{diag}(t_1, t_2, -t_2, -t_2)$ in the Lie algebra of Sp(4) corresponds to $\mathrm{diag}(t_1 + t_2, t_1 - t_2, 0, t_2 - t_1, -t_1 - t_2)/2$ in the Lie algebra of SO(5).

A formula for the Shalika germ of the two-regular unipotent class in Sp(4) was obtained in [H1]. Let us recall the formula. Let X_1 be the variety obtained from $X_0 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by blowing up the diagonal $\mathbb{P}^1 \simeq \{(x, x, x, x)\} \subset X_0$. The cross-ratio of four distinct points in $\mathbb{P}^1$ with affine coordinates $x_1, \ldots, x_4$ is defined as

$$c(x_1, \ldots, x_4) = \frac{(x_2 - x_4)(x_1 - x_3)}{(x_3 - x_4)(x_1 - x_2)}.$$

(We depart from the standard normalization, which takes the cross-ratio to be $c/(c-1) = (x_1 - x_3)(x_2 - x_4)/((x_1 - x_4)(x_2 - x_3))$.) We pull the cross-ratio back, without change of notation, to a Zariski open set of X_1. We define a double cover X of X_1 by extracting a square root of the cross-ratio: $c_1^2 = c(x_1, \ldots, x_4)$. For any two constants t_1, t_2 in $\bar{F}$, we consider the meromorphic differential form ω on X:

$$\omega = \frac{t_1 t_2 (t_2^2 - t_1^2) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{2c_1(-t_1 + c_1 t_2)^2 (-x_1 + x_2)^2 (-x_3 + x_4)^2}.$$

It is easily checked that ω is invariant under the group of linear fractional transformations of X:

$$x_i \mapsto \frac{a_{11} x_i + a_{12}}{a_{21} x_i + a_{22}}, \quad c_1 \mapsto c_1, \quad i = 1, 2, 3, 4.$$

We consider the Coxeter group

$$W = \langle \sigma_\alpha, \sigma_\beta : \sigma_\alpha^2 = \sigma_\beta^2 = (\sigma_\alpha \sigma_\beta)^4 = 1 \rangle.$$

The group W acts on the elements $\{\pm t_1, \pm t_2\}$ by $\sigma_\alpha(\pm t_1) = \pm t_2$, $\sigma_\beta(\pm t_1) = \pm t_1$, and $\sigma_\beta(\pm t_2) = \mp t_2$. The group W acts on the coordinates of X_0 and X_1 by

$$\sigma_\alpha(x_1) = x_1, \quad \sigma_\beta(x_1) = x_2,$$
$$\sigma_\alpha(x_2) = x_3, \quad \sigma_\beta(x_2) = x_1,$$
$$\sigma_\alpha(x_3) = x_2, \quad \sigma_\beta(x_3) = x_4,$$
$$\sigma_\alpha(x_4) = x_4, \quad \sigma_\beta(x_4) = x_3.$$

This implies that $\sigma_\beta(c) = c$ and $\sigma_\alpha(c)c = 1$, where c is the cross-ratio of $x_1, \ldots, x_4$. We lift the action of W to X by the conditions $\sigma_\alpha(c_1)c_1 = 1$ and $\sigma_\beta(c_1) = -c_1$, where $c_1^2 = c$.

The variety X is naturally defined over F, by taking the coordinates $x_1, \ldots, x_4$, and c_1 to be defined over F. We use the action of W to define a twisted form of the variety X. Assume that we are given a homomorphism $\phi : \mathrm{Gal}(\bar{F}/F) \to W$. We also assume that t_1 and t_2 are parameters in $\bar{F}$ that satisfy $t_2^2 \neq t_1^2$ and $\sigma(t_i) = \phi_\sigma(t_i)$, for $i = 1, 2$ and $\sigma \in \mathrm{Gal}(\bar{F}/F)$. Considering ϕ as a cocycle of $\mathrm{Gal}(\bar{F}/F)$ with values in the group of automorphisms of X, we twist the Galois structure by ϕ. Let $X^*(F)$ denote the set of F-points of the variety X for this twisted structure. We identify W with the Weyl group of Sp(4), making σ_α the reflection through a short simple root α.

Lemma 3.1. *The two-regular stable Shalika germ on Sp(4), attached to the parameter*

$$diag(t_1, t_2, -t_2, -t_1) \in \mathfrak{sp}(4)$$

(in the stable conjugacy class attached to the map $\phi : Gal(\bar{F}/F) \to W$) is equal to the principal-value integral

$$\int_{X^*(F)} |\omega|.$$

Proof. The lemma holds even if the residual characteristic of the p-adic field F is even. This is a reformulation of [H1]. In that paper we showed that each Borel subgroup in the *star* of Borel subgroups ($B(W)$) lies in unique projective line ℓ_β of type β in the variety of Borel subgroups containing a given two-regular unipotent element. Since the coordinate $z(\alpha)$ is equal to zero on the divisor corresponding to the two-regular class, we have $B(W) = B(W')$ whenever the wall between the Weyl chambers W and W' is of type α. The eight Borel subgroups are thus arranged into four pairs. Thus there is a morphism of varieties from the variety Y_Γ computing the Shalika germs to the fourfold product of $\mathbb{P}^1$, sending each point to the associated four points ($B(W)$) in ℓ_β. The construction in [La,§3] (see also [H2,I§4]) blows up along a certain subvariety in the variety of stars, and this corresponds in our picture to blowing up along the diagonal subvariety. To see that the map to the fourfold product of projective lines is of degree two, we observe that by fixing the four Borel subgroups ($B(W)$), we are fixing the local coordinates $r_1, \ldots, r_4$ of [H1,§7]. It is clear from the relations between the coordinates s_i and r_i of [H1,Equation 7.12] that generically there are two stars for each choice of parameters r_i. Thus we have morphism of degree two.

We note that the morphism from the four-dimensional variety Y_Γ to the double cover of X is birational but not an isomorphism. One way to see this is to note that in [H1,p.254] we showed that the square root of the cross-ratio (which up to a linear fractional transformation is the coordinate we denoted ℓ in [H1]) gives a well-defined morphism $Y \to \mathbb{P}^1$. But it is easy to check that the cross-ratio does not extend to all of X. For instance, it does not extend to points $x_1 = x_2 = x_3 \neq x_4$. (The morphism Y_Γ restricts to an isomorphism on the complement of the the *spurious divisors* in [H1].)

The preceding comments show that the stable Shalika germ associated with the two-regular unipotent class should have the general shape given in the lemma. To make the relationship precise we need to express the local coordinates w, ℓ, z, and ξ of the paper [H1,§6] in terms of the local coordinates $x_1, \ldots, x_4$ and c_1 introduced above. The explicit transformation is given by $\xi = x_1$, $\ell = (-t_2c_1 + t_1)/(t_2c_1(t_2^2 - t_1^2))$, $z = (x_2 - x_1)/(2t_2)$, and

$$-2t_2w + 1 = \frac{(t_1 + t_2)c_1(c_1 - 1)(x_1 - x_2)(x_4 - x_3)}{(-t_1 + t_2)(-x_1 + x_3)(-x_1 + x_4)}.$$

The differential form is given in [H1,p.240] as

$$\omega = \frac{dw \wedge d\ell \wedge dz \wedge d\xi}{(-2t_2w + 1)\ell^2 z^2}.$$

A lengthy computation of the Jacobian of the transformation from one coordinate system to the other shows that the form ω transforms to the differential form given in the statement of the lemma. The Galois action on ℓ given in [H1] determines the Galois action on the coordinate c_1. The Galois action on $x_1, \ldots, x_4$ may be obtained from the explicit transformation given above, or it may be obtained directly from the description of the twisted Galois action on Y_Γ given in [La,§2], which states that the Galois group acts on a star $(B(W))$ by permuting the chambers W according to the Weyl group. This completes the reformulation of the stable two-regular Shalika germ. □

We classify the possible homomorphisms $\phi : \mathrm{Gal}(\bar{F}/F) \to W$. Let K be the Galois extension of F fixed by the kernel of ϕ. Then $\mathrm{Gal}(K/F)$ is isomorphic to $\mathrm{Im}(\phi)$. The ramification is tame because the order of W and the residue characteristic of F are relatively prime. If W' is a subgroup of W, let $K^{W'}$ denote the subfield of K fixed by the subgroup $W' \cap \mathrm{Im}(\phi)$ of $\mathrm{Gal}(K/F)$. Let $\langle\sigma_1, \ldots, \sigma_\ell\rangle$ denote the subgroup of W generated by $\sigma_1, \ldots, \sigma_\ell$.

Lemma 3.2. *It is enough to consider the following homomorphisms ϕ.*
(K biquadratic)

(1) Im(ϕ) = $\langle\sigma_\beta, \sigma_\alpha\sigma_\beta\sigma_\alpha\rangle$, the fields $K^{\langle\sigma_\beta\rangle}$ and $K^{\langle\sigma_\alpha\sigma_\beta\sigma_\alpha\rangle}$ are ramified quadratic extensions of F.
(2) Im(ϕ) = $\langle\sigma_\beta, \sigma_\alpha\sigma_\beta\sigma_\alpha\rangle$, $K^{\langle\sigma_\beta\rangle}$ is ramified, and $K^{\langle\sigma_\alpha\sigma_\beta\sigma_\alpha\rangle}$ is unramified.

(K cyclic of order four)

(3) Im(ϕ) = $\langle\sigma_\alpha\sigma_\beta\rangle$, and K is an unramified cyclic extension of degree four.
(4) Im(ϕ) = $\langle\sigma_\alpha\sigma_\beta\rangle$, K is a totally ramified cyclic extension of degree four, and -1 is a square in F.
(5) Im(ϕ) = $\langle\sigma_\alpha\sigma_\beta\rangle$, and the maximal unramified subfield $F \subset E \subset K$ has degree two over F.

(K dihedral)

(6) Im(ϕ) = W, the subfield $E := K^{\langle\sigma_\alpha\sigma_\beta\rangle}$ is an unramified extension of F of degree two, and K/E is a totally ramified extension of degree four. Also -1 is not a square in F.

Proof. If the homomorphism ϕ does not define an elliptic Cartan subgroup, then the (stable or unstable) two-regular Shalika germ in Sp(4) is identically zero on that Cartan subgroup. Thus we may exclude ϕ if the image of ϕ is trivial or conjugate to a subgroup of order two generated by a simple reflection. If the image of ϕ is contained in $\langle\sigma_\alpha, \sigma_\beta\sigma_\alpha\sigma_\beta\rangle$, then the germs were studied in full in [H1,§6]. In particular, it was shown that the stable two-regular Shalika germ vanishes identically on the Cartan subgroups attached to such ϕ. The remaining subgroups of W are $\langle\sigma_\beta, \sigma_\alpha\sigma_\beta\sigma_\alpha\rangle$ (biquadratic), $\langle\sigma_\alpha\sigma_\beta\rangle$ (cyclic), and W (dihedral). There is a unique biquadratic extension K of F, and cases 1 and 2 enumerate the possible positions of the unramified degree-two subfield of K. Two homomorphisms ϕ and ϕ' define the same stable conjugacy class of Cartan subgroups if they are conjugate: $\phi' = w\phi w^{-1}$, for some $w \in W$. Thus, if $K^{\langle\sigma_\beta\rangle}$ is unramified, we conjugate by $\sigma_\alpha \in W$ to bring us back to Case 2.

The subgroup $\langle\sigma_\alpha\sigma_\beta\rangle$ of W is cyclic of order four (C_4). By the local class field theory of tame abelian extensions, we may replace the domain of ϕ by $k^\times \times \mathbb{Z}$, where k is the residue field of F. We run through the homomorphisms $k^\times/k^{\times 4} \times \mathbb{Z}/4\mathbb{Z} \to C_4$. If 4 divides $q-1$, then $k^\times/k^{\times 4}$ is cyclic of order four, so we consider homomorphisms $C_4 \times C_4 \twoheadrightarrow C_4$. There are six possible kernels, and six possible fields K. One is unramified, one has a degree two unramified subfield, and four are totally ramified. If 4 does not divide $q-1$, then $k^\times/k^{\times 4}$ is cyclic of order two, and there are two possible kernels to homomorphisms $C_2 \times C_4 \twoheadrightarrow C_4$. One gives an unramified extension, the other an extension with an unramified subfield of degree two. This gives Cases 3, 4, and 5.

Now assume that the image of ϕ is W. The Galois group $\mathrm{Gal}(K/F)$, hence W, has a normal cyclic subgroup W_0 such that W/W_0 is the cyclic Galois group of the maximal unramified extension of F in K (see [I,2.5]). The only nontrivial normal cyclic subgroups of W are $\langle\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\rangle$ and $\langle\sigma_\alpha\sigma_\beta\rangle$, and only $\langle\sigma_\alpha\sigma_\beta\rangle$ gives a cyclic quotient. So the maximal unramified extension is $K^{\langle\sigma_\alpha\sigma_\beta\rangle}$.

We show that -1 is not a square in F. Set $K_1 = K^{\langle\sigma_\beta,\sigma_\alpha\sigma_\beta\sigma_\alpha\rangle}$. Write $K^{\langle\sigma_\beta\rangle} = K_1(t_1)$, for some t_1 satisfying $t_1^2 \in K_1$. Set $u = t_1/\sigma_\alpha(t_1)$. Then $\sigma_\beta(u) = -u$ and $\sigma_\alpha(u)u = 1$. We have the following congruences modulo the maximal ideal in O_K:

$$-1 \equiv \sigma_\alpha\sigma_\beta(u)u \equiv u^2, \qquad \pm\sqrt{-1} \equiv u,$$
$$1 \equiv \sigma_\alpha(u)u \equiv \sigma_\alpha(\sqrt{-1})\sqrt{-1}.$$

So $\sigma_\alpha(\sqrt{-1}) = -\sqrt{-1}$, and $\sqrt{-1} \notin F$. □

The following special case of a general conjecture of Assem and Kottwitz will complete the proof of Theorem 1.1.

Theorem 3.3. $\int_{X^*(F)} |\omega| = 0$, *for any p-adic field F of characteristic zero of odd residual characteristic.*

Proof. By the lemma, there are six cases to consider. In each of these cases $\mathrm{Im}(\phi)$ acts transitively on the coordinates x_i. Thus $|x_i| = |x_j|$ for all i, j on the set of F-rational points. For each $x_1 \in \mathbb{P}^1$, we have $|x_1| \le 1$ or $|x_1| > 1$. We have already remarked that the group of linear fractional transformations acts on X preserving the form ω. Replacing x_i by $1/x_i$ when $|x_1| > 1$, we see that $X^*(F)$ is the union of two sets $P_1 = \{(x_1, \ldots, x_4, c_1) : \forall i,\ |x_i| \le 1\}$ and $P_2 = \{(x_1, \ldots, x_4, c_1) : \forall i,\ |x_i| < 1\}$. We will show that the integral of ω over the union of these two sets is zero. The rough idea is to produce a morphism to a projective line $\mathbb{P}^1$ (essentially given by $(x_1, \ldots, x_4, c_1) \mapsto c_1$), to check that the volumes of the fibers of this map are all equal, and to verify that the integral over the projective line is zero. The integral over the projective line will reduce to $\int_{\mathbb{P}^1} |dx/x^2|$. We then appeal to Lemma 2.2 with $\theta(x) = 1/|x|$.

Two-regular Cases 1 and 2. Biquadratic Extensions.

Let us turn to the proof of Theorem 3.3 in Cases 1 and 2 (K biquadratic). We identify $\mathrm{Gal}(K/F)$ with $\langle\sigma_\beta, \sigma_\alpha\sigma_\beta\sigma_\alpha\rangle$ by ϕ, and we set $\sigma'_\beta = \sigma_\alpha\sigma_\beta\sigma_\alpha$. The Galois group acts on the coordinates $x_1, \ldots, x_4$ by the relations:

$$\begin{array}{ccc} x_1 & \xrightarrow{\sigma'_\beta} & x_3 \\ {\scriptstyle\sigma_\beta}\downarrow & & \downarrow{\scriptstyle\sigma_\beta} \\ x_2 & \xrightarrow[\sigma'_\beta]{} & x_4 \end{array}$$

We write $K = F(e, e')$, where $\sigma_\beta(e) = -e$, $\sigma_\beta(e') = e'$, $\sigma'_\beta(e) = e$, and $\sigma'_\beta(e') = -e'$. We assume that e is a unit or, in the ramified case, a uniformizing element in $F(e)$. In Cases 1 and 2, the field $F(e') = K^{\langle\sigma_\beta\rangle}$ is ramified, and we take e' to be a uniformizer in $F(e')$. We fix F-coordinates u_i by the relation

$$x_1 = u_0 + u_1 e + u_2 e' + u_3 ee'.$$

Then also

$$\begin{aligned} x_2 &= u_0 - u_1 e + u_2 e' - u_3 ee', \\ x_3 &= u_0 + u_1 e - u_2 e' - u_3 ee', \\ x_4 &= u_0 - u_1 e - u_2 e' + u_3 ee'. \end{aligned}$$

The cross-ratio $c = c(x_1, \ldots, x_4)$ becomes

$$c = \frac{e'^2(u_2^2 - u_3^2 e^2)}{e^2(u_1^2 - u_3^2 e'^2)}.$$

By the conditions of Lemma 3.1, its square root c_1 must satisfy $\sigma_\beta(c_1) = -c_1$ and $\sigma'_\beta(c_1) = -c_1$. Thus $(u_2^2 - u_3^2 e^2)/(u_1^2 - u_3^2 e'^2)$ is a square in F. This is the quotient of norms from two distinct quadratic extensions of F, so it is a square only if both $u_2^2 - u_3^2 e^2$ and $u_1^2 - u_3^2 e'^2$ are squares. Since e' is a uniformizer in a ramified extension, and $u_1^2 - u_3^2 e'^2$ is a square in F, we see that $|u_3| \le |u_1|$.

Instead of the sets P_1 and P_2, we may integrate over

$$P'_1 = \{(u_0, \ldots, u_3, v) \in F^5 : |u_0| \le 1,\ |u_1| \le 1,\ |u_2| \le 1,\ |u_3| \le |u_1|,\ v^2 = u_2^2 - u_3^2 e^2\}$$

and

$$P'_2 = \{(u_0, \ldots, u_3, v) \in F^5 : |u_0| < 1,\ |u_1 e| < 1,\ |u_2| \le 1,\ |u_3| \le |u_1|,\ v^2 = u_2^2 - u_3^2 e^2\}.$$

To construct X we blow up along the diagonal, expressed in these coordinates by the equations $u_1 = u_2 = u_3 = 0$. The blowing-up is defined by the equations $u_i U_j = u_j U_i$, for $i, j = 1, 2, 3$, where (U_1, U_2, U_3) are homogeneous coordinates on $\mathbb{P}^2$. Each such point satisfies exactly one of the following three pairs of constraints depending on which of the coordinates U_i is largest:

$$\begin{aligned} Q_1 :&\quad |U_1| \ge |U_2|,\ |U_1| \ge |U_3|, \\ Q_2 :&\quad |U_2| > |U_1|,\ |U_2| \ge |U_3|, \\ Q_3 :&\quad |U_3| > |U_1|,\ |U_3| > |U_2|. \end{aligned}$$

The third possibility Q_3 never actually arises because the condition $|u_3| \le |u_1|$ implies $|U_3| \le |U_1|$. We set $u_{ij} = U_i/U_j$.

On Q_1 we use local coordinates c_1, u_0, u_1, and u_{32}. The coordinates u_0, u_1, u_{32}, and v are defined over F, and the coordinate c_1 lies in the imaginary axis of $K^{\langle\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\rangle}$. Expressed in local coordinates, we have

$$\begin{aligned} Q_1 \cap P'_1 &= \{(c_1, u_0, u_1, u_{32}, v) : |c_1| \le 1,\ |u_0| \le 1,\ |u_1| \le 1,\ v^2 = 1 - e^2 u_{32}^2\}, \\ Q_1 \cap P'_2 &= \{(c_1, u_0, u_1, u_{32}, v) : |c_1| \le 1,\ |u_0| < 1,\ |u_1 e| < 1,\ v^2 = 1 - e^2 u_{32}^2\}, \\ |\omega| &= \left| \frac{t_1 t_2 (t_2^2 - t_1^2) du_0\, du_1\, dc_1\, du_{32}}{u_1^2(-t_1 + t_2 c_1)^2 (1 - e^2 u_{32}^2)} \right|. \end{aligned}$$

In carrying out this calculation, we found it convenient first to express $Q_1 \cap P'_1$, $Q_1 \cap P'_2$, and $|\omega|$ in terms of the local coordinates u_0, u_1, u_{21}, and u_{32}. Each choice of u_0, u_1, u_{21}, and u_{32} for which $1 - e^2 u_{32}^2$ is a square in F leads to two points on the double cover defined by $c_1^2 = c$, and, similarly, given (c_1, u_0, u_1, u_{32}) there are two choices of u_{21} obtained by solving a quadratic equation. The quadratic equation has roots in F if $1 - e^2 u_{32}^2$ is a square in F.

These sets are Cartesian products and the measure is also a product. Thus it is an elementary matter to integrate over the coordinates u_0, u_1, and u_{32}. We find that the combined contribution from $Q_1 \cap P_1'$ and $Q_1 \cap P_2'$ is

$$-2\left(\frac{1}{q} + \frac{1}{q^2}\right)|t_1 t_2(t_2^2 - t_1^2)| \int_{|c_1|\le 1} \left|\frac{dc_1}{(-t_1 + t_2 c_1)^2}\right|.$$

On Q_2 we use local coordinates c_1, u_0, u_2, and u_{31}. Expressed in local coordinates we have

$$Q_2 \cap P_1' = \{(c_1, u_0, u_2, u_{31}, v) : |c_1| > 1,\ |u_0| \le 1,\ |u_2| \le 1,\ |u_{31}| \le 1,\ v = \pm 1\},$$
$$Q_2 \cap P_2' = \{(c_1, u_0, u_2, u_{31}, v) : |c_1| > 1,\ |u_0| < 1,\ |u_2| \le 1,\ |u_{31}| \le 1,\ v = \pm 1\},$$
$$|\omega| = \left|\frac{t_1 t_2 (t_2^2 - t_1^2) du_0\, du_2\, dc_1\, du_{31}}{u_2^2(-t_1 + t_2 c_1)^2}\right|.$$

We integrate over the coordinates u_0, u_2, and u_{31}, and find that the contribution from $Q_2 \cap P_1'$ and $Q_2 \cap P_2'$ is

$$-2\left(\frac{1}{q} + \frac{1}{q^2}\right)|t_1 t_2(t_2^2 - t_1^2)| \int_{|c_1|> 1} \left|\frac{dc_1}{(-t_1 + t_2 c_1)^2}\right|.$$

Thus the stable two-regular germ is

$$-2\left(\frac{1}{q} + \frac{1}{q^2}\right)|t_1 t_2(t_2^2 - t_1^2)| \int_{\mathbb{P}^1} \left|\frac{dc_1}{(-t_1 + t_2 c_1)^2}\right|. \tag{3.4}$$

Under the change of coordinates $x = 1 - t_2 c_1/t_1$ the integral over $\mathbb{P}^1$ reduces to the constant $|t_1 t_2|^{-1}$ times $\int_{\mathbb{P}^1} |dx/x^2| = 0$. Thus the stable two-regular germ vanishes in Cases 1 and 2.

Two-regular Case 3. Unramified Cyclic Extensions.

We begin with some comments that apply to all of the remaining cases. Set

$$y_1 = x_3 - x_4,$$
$$y_2 = x_1 - x_3,$$
$$y_3 = x_2 - x_1,$$
$$y_4 = x_4 - x_2.$$

Then $y_1 + y_2 + y_3 + y_4 = 0$, and the cross-ratio becomes $c = c_1^2 = y_4 y_2/(y_1 y_3)$. The Weyl-group action on the coordinates x_i becomes

$$\begin{aligned} \sigma_\beta\sigma_\alpha(y_1) &= y_2, & \sigma_\beta(y_1) &= -y_1,\\ \sigma_\beta\sigma_\alpha(y_2) &= y_3, & \sigma_\beta(y_2) &= -y_4,\\ \sigma_\beta\sigma_\alpha(y_3) &= y_4, & \sigma_\beta(y_3) &= -y_3,\\ \sigma_\beta\sigma_\alpha(y_4) &= y_1, & \sigma_\beta(y_4) &= -y_2. \end{aligned} \tag{3.5}$$

In Cases 3,4,5, and 6, the Galois group $\mathrm{Gal}\,(K/F)$ acts transitively on the coordinates y_1, y_2, y_3, y_4, so $|y_i| = |y_j|$, for all i and j, and c_1 is a unit in the extension $K^{\langle\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\rangle}$. Set

$\omega' = y_1^{-3} dy_1 \wedge dy_2 \wedge dy_3/(-1+tc_1)^2$, where t is defined to be t_2/t_1. For $\ell \in \mathbb{Q}$, let Y_ℓ denote the set

$$\begin{aligned} Y_\ell = \{(y_1, y_2, y_3, y_4, c_1) \in K^5 :\ & y_1 + y_2 + y_3 + y_4 = 0,\ |y_i|^4 = q^{-\ell},\ \sigma(y_i) = \phi_\sigma(y_i), \\ & c_1^2 = y_2 y_4/(y_1 y_3),\ \ \sigma(c_1) = \phi_\sigma(c_1), \\ & \text{for } i = 1,2,3,4, \text{ and } \sigma \in \mathrm{Gal}\,(K/F)\}. \end{aligned}$$

Let Z_ℓ denote the integral $\int_{Y_\ell} |\omega'|$.

Lemma 3.6. *In the context of Cases 3,4,5, and 6, suppose that $Z_\ell = 0$, for $\ell = 0, 1, 2, 3$. Then the stable two-regular germ is zero. In fact,*

$$\frac{1}{|t_2^2 - t_1^2|} \int_{X^*(F)} |\omega| = \frac{2}{1-q} Z_0 + \frac{q^{1/4}(q+1)}{q(1-q)} Z_1 + \frac{q^{1/2}(q+1)}{q(1-q)} Z_2 + \frac{q^{3/4}(q+1)}{q(1-q)} Z_3.$$

Proof. The ramification index $e(K/F)$ of the field extension K/F divides four, and $Y_\ell = \emptyset$ unless $e(K/F)\cdot \ell \in 4\mathbb{Z}$. Thus $Z_\ell = 0$, unless $\ell \in \mathbb{Z}$. Also, if π is a uniformizer in F, then $y_i \mapsto \pi y_i$ is a form-preserving bijection between Y_ℓ and $Y_{\ell+4}$ so that $Z_\ell = Z_{\ell+4}$.

On the variety X, we introduce coordinates $y_1 = x_3 - x_4$, and so forth, as above. As we remarked, the rational points of X^* will satisfy $|y_i| = |y_j|$. By introducing the coordinates $x_1' = (x_1+x_2+x_3+x_4)/4, y_1, y_2, y_3$ on the patches P_1 and P_2 described above, the differential form on X^* becomes

$$\omega = \frac{-dx_1' \wedge dy_1 \wedge dy_2 \wedge dy_3\, t(t_2^2 - t_1^2)}{2 y_3^2 y_1^2 (-1 + c_1 t)^2 c_1}.$$

We may ignore the unit $t/(2c_1)$ in the form because it does not affect the measure. The coordinate x_1' is defined over F, and satisfies $|x_1'| \le 1$ on P_1 and $|x_1'| < 1$ on P_2. We integrate over x_1' to obtain a form independent of x_1'. The remaining contribution to the stable two-regular germ is the analytic continuation of

$$|t_2^2 - t_1^2| \sum_\ell q^{-s+\ell/4} \int_{Y_\ell} |\omega'|, \ \ \mathrm{Re}(s) \gg 0,$$

to $s = 0$. The sum runs over $\{\ell \in \mathbb{Z} : \ell \ge 0\}$ and $\{\ell \in \mathbb{Z} : \ell > 0\}$, for P_1 and P_2. Summing the series, we obtain Lemma 3.6. Since $|y_i| = |y_j|$ on the set of rational points near the diagonal $y_1 = y_2 = y_3 = 0$, the rational points are concentrated in a particular direction, and blowing-up has no effect on the principal-value integral. □

Now in Case 3, the extension K/F is cyclic and unramified. Set $\sigma = \sigma_\beta \sigma_\alpha$, and $E = K^{\langle \sigma^2 \rangle}$. We have $c_1 \in E$, and $\sigma(c_1)c_1 = -1$. We let k_4, k_2, and k denote the residue fields of K, E, and F. We let $\mathrm{Gal}\,(K/F)$ act on k_4 and k_2 without change in notation. In the following cases, we let $\mathfrak{p}$ denote the maximal ideal of O_K. Since K/F is unramified, it is enough to verify that $Z_0 = 0$ in Lemma 3.6. Fix an element $\xi \in k_2$ with $\sigma(\xi)\xi = -1$. Writing $\sigma(\xi) = \xi^q$, we see that ξ must be one of the $q+1$ distinct roots of $\xi^{q+1} + 1 = 0$. Consider the set $R(\xi) \subset k_4^5$ consisting of reductions modulo $\mathfrak{p}$ of elements $(y_1, \ldots, y_4, c_1)$ in Y_0 for which c_1 reduces to ξ. We will show below that the cardinality of $R(\xi)$ is a constant r independent of ξ. Assume this result for a moment. We break the integral over Y_0 into sums over ξ and $R(\xi)$ according to the residue class of the element of Y_0. Following Denef [D], for each term of the sum we pick local coordinates g_1, g_2, and g_3 with $|g_i| < 1$, for

$i = 1, 2, 3$. The measure $|\omega'|$ becomes $|dg_1\, dg_2\, dg_3|$ if $(-1+tc_1) \not\equiv 0 \mod \mathfrak{p}$, and it becomes $|g_1^{-2} dg_1\, dg_2\, dg_3|$ otherwise. Then the integral of $|\omega'|$ over Y_0 is

$$(3) \qquad \sum_{\substack{\xi \\ -1+\xi t \not\equiv 0}} \sum_{R(\xi)} \int |dg_1\, dg_2\, dg_3| + \sum_{\substack{\xi \\ -1+\xi t \equiv 0}} \sum_{R(\xi)} \int |g_1^{-2} dg_1\, dg_2\, dg_3|.7$$

$$= q \cdot r \cdot \frac{1}{q^3} + 1 \cdot r \cdot \frac{-1}{q^2} = 0.$$

It remains to be seen that the cardinality of $R(\xi)$ is independent of ξ. For this we work entirely over the finite fields k_4, k_2, and k, taking y_i to be coordinates in k_4. Fix ξ in k_2 with $\sigma(\xi)\xi = -1$, and fix an element $e \in k_2$ such that $\sigma(e) = -e$. We count nonzero elements $y \in k_4$ whose trace in k is zero such that $\xi^2 = \sigma(y)\sigma^3(y)/(y\sigma^2(y))$. Since ξ^2 has norm one, we may select $\xi_1 \in k_2^\times$ such that $\xi^2 = \sigma(\xi_1)/\xi_1$. Then for every $z \in k^\times$ and $u \in ek$ for which $u^2 - 4\xi_1 z$ is not a square in k_2, the equation

$$y^2 - uy + z\xi_1 = 0$$

provides two distinct solutions y, and every solution y appears as the root of a unique quadratic equation of this form. (If $u^2 - 4\xi_1 z$ is a square v^2, for some $v \in k_2$, then $y \in k_2$, and so $\xi = \pm\sigma(y)/y$, and $\sigma(\xi)\xi = 1$, contrary to hypothesis.)

It suffices to show that the number of solutions to the complementary condition

$$u^2 - 4\xi_1 z = v^2,\ u \in ek,\ v \in k_2,\ z \in k^\times$$

is independent of ξ_1. Setting $a = u - v \in k_2^\times$ and $b\,z = u + v \in k_2^\times$, we find that $b = 4\xi_1/a$. Moreover, $z \in k^\times$ is uniquely determined in terms of a by the condition

$$\mathrm{trace}_{k_2/k}(a + zb) = \mathrm{trace}_{k_2/k} 2u = 0$$

whenever $a \notin \xi_1 ek^\times$, $ek^\times$ (that is, the traces of a and b are nonzero). The lines $\xi_1 ek$ and ek are distinct: if $\xi_1 \in k^\times$, then $\sigma(\xi)\xi = 1$, contrary to hypothesis. If $a \in ek$ or $a \in \xi_1 ek$, then we would obtain the contradiction $z = 0$ or $\mathrm{trace}_{k_2/k} u \neq 0$. The solutions are thus parametrized by the element a, which lies in the complement of two intersecting lines in the plane k_2, a set of cardinality $q^2 - 2q + 1$. The cardinality r of the original set $R(\xi)$ is $2(q^2 - q) - (q^2 - 2q + 1) = q^2 - 1$. This completes the proof for Case 3.

Two-regular Case 4. Totally Ramified Cyclic Extensions.

We assume that K/F is a totally ramified cyclic extension of degree four. Let E/F be the quadratic extension of F in K. Lemma 3.2 shows that -1 is a square in F. Fix a square root i of -1. Fix a uniformizer π' of K and a generator σ of the cyclic Galois group $\mathrm{Gal}\,(K/F)$ such that $\sigma(\pi') = i\pi'$. If $y_1 = \pi'^{\ell} y_1'$, where y_1' is a unit, then $\sigma(y_1') \equiv y_1'$ modulo the maximal ideal $\mathfrak{p}$ of O_K, so

$$(3.8) \qquad -1 \equiv \sigma(c_1)c_1 \equiv c_1^2 \equiv c \equiv \frac{y_2 y_4}{y_1 y_3} \equiv \frac{\sigma(y_1)\sigma^3(y_1)}{y_1\sigma^2(y_1)} \equiv \frac{i^\ell \cdot i^{3\ell}}{1 \cdot i^{2\ell}} \equiv i^{2\ell} \equiv (-1)^\ell \mod \mathfrak{p}.$$

So ℓ is odd. Conversely, if ℓ is odd, then the cross-ratio c has two square roots c_1 satisfying $\sigma(c_1)c_1 = -1$. We will treat the case $\ell = 1$, the case $\ell = 3$ being similar.

Recall that $t := t_2/t_1$. It satisfies $t^2 \in E$, and $t^2 = -1 + a\pi'^2$, for some $a \in O_E$. We write the element y_1 of trace zero as $y_1 = \pi' u_1 + \pi'^2 u_2 + \pi'^3 u_3$, with $u_2, u_3 \in O_F$, and $u_1 \in O_F^\times$. We set $x = (-1 + c_1 t)/(\pi'^2(1 + c_1 t)) \in F$. Then $|-1 + c_1 t| < 1$ if and only if $|x| \leq 1$. For any $u_1 \in O_F^\times$, $u_2, u_3 \in O_F$, there are two choices of square root c_1. For one $|-1 + c_1 t| < 1$, and for the other $|-1+c_1t| = 1$. We switch to the local F-coordinates x, u_1, and u_2 when $|-1+c_1t| < 1$. Expressed in terms of these coordinates we have $Y_1 = Y' \cup Y''$, where

$$\begin{aligned} Y' &= \{(u_1,u_2,x) : \ |u_1| = 1,\ |u_2| \le 1,\ |x| \le 1\}, \\ Y'' &= \{(u_1,u_2,u_3) : |u_1| = 1,\ |u_2| \le 1,\ |u_3| \le 1\}, \\ |\omega'| &= q^{1/4}\left|\frac{du_1\, du_2\, dx}{x^2}\right| \text{ on } Y', \\ &= q^{-3/4}\,|du_1\, du_2\, du_3| \text{ on } Y''. \end{aligned}$$

The integral over Y_1 is then

$$Z_1 = q^{1/4}\int_{Y'}\left|\frac{du_1\, du_2\, dx}{x^2}\right| + q^{-3/4}\int_{Y''}|du_1\, du_2\, du_3|\,. \tag{3.9}$$

In the second integral we set $u_3 = 1/(\pi x)$. We then see that the integral reduces to

$$Z_1 = q^{1/4}\int_{\substack{|u_1|=1\\|u_2|\le 1}}|du_1\, du_2|\int_{\mathbb{P}^1}\left|\frac{dx}{x^2}\right| = q^{1/4}\left(1-\frac{1}{q}\right)\int_{\mathbb{P}^1}\left|\frac{dx}{x^2}\right|.$$

If $\ell = 3$, write $y_1 = \pi'^3 u_1 + \pi'^5 u_2 + \pi'^6 u_3$, with $u_1 \in O_F^\times$ and $u_2, u_3 \in O_F$. A similar argument shows

$$Z_3 = q^{-1/4}\int|du_1\, du_3|\int_{\mathbb{P}^1}\left|\frac{dx}{x^2}\right| = q^{-1/4}\left(1-\frac{1}{q}\right)\int_{\mathbb{P}^1}\left|\frac{dx}{x^2}\right|. \tag{3.10}$$

By Lemma 2.2, these integrals are zero. This completes Case 4.

Two-regular Case 5. Ramified Cyclic Extensions.

We assume that K/F is cyclic of degree four, and that the maximal unramified extension in K is a field E of degree two over F. We pick a uniformizer π' in K satisfying $\pi'^2 = \pi\beta$, where π is a uniformizer in F and $\beta \in E$ is not a square. Write $y_1 = \pi'^m y_1'$, where y_1' is a unit and $2m = \ell$. Set $\gamma = \sigma(\pi')/\pi'$, where σ is a fixed generator of $\mathrm{Gal}(K/F)$. We find that $\sigma(\gamma)\gamma = -1$. Set $z = (y_1'\sigma^2(y_1'))^{1/2}$. Since K/E is ramified, z belongs to E. Then $c_1 = \pm\gamma^m\sigma(z)/z$, and $-1 \equiv \sigma(c_1)c_1 \equiv (-1)^m$ modulo $\mathfrak{p}$, the maximal ideal of O_K. So m is odd. Conversely if m is odd, the two square roots $c_1 = \pm\gamma^m\sigma(z)/z$ satisfy $\sigma(c_1)c_1 = -1$. We may take $m = 1$ ($\ell = 2$), since ℓ lies in the range $0, 1, 2, 3$. Fix $e \in O_E^\times$ with $\sigma(e) = -e$.

We write trace-zero elements as

$$y_1 = \pi'(u_1 + eu_2) + \pi e u_3, \quad (u_1 + eu_2) \in O_E^\times,\ u_3 \in O_F.$$

This gives O_F-coordinates, and we may reduce to the finite field as in Case 3. We have $c_1 \equiv \pm\gamma(u_1 - eu_2)/(u_1 + eu_2) \mod \mathfrak{p}$. For a given residue class ξ for c_1, the cardinality of the set $R(\xi)$ of residual solutions $u_1 + eu_2 \in O_E^\times$, $u_3 \in O_F$ to the equation $\xi \equiv \pm\gamma(u_1+eu_2)/(u_1-eu_2)$ is $r = 2q(q-1)$, which is independent of ξ. As in Case 3, the stable two-regular germ is zero. In fact, the measure is

$$|\omega'| = q^{-1/2}|du_1\, du_2\, du_3(-1+c_1t)^{-2}|,$$

and the integral of $|\omega'|$ over Y_2 is

$$\begin{aligned} Z_2 &= q^{-1/2}\sum_{\substack{\xi\\-1+t\xi\not\equiv 0}}\sum_{R(\xi)}\int|dg_1\, dg_2\, dg_3| + q^{-1/2}\sum_{\substack{\xi\\-1+t\xi\equiv 0}}\sum_{R(\xi)}\int\left|\frac{dg_1}{g_1^2}dg_2\, dg_3\right| \\ &= q^{-1/2}\cdot q\cdot r\cdot\frac{1}{q^3} + q^{-1/2}\cdot 1\cdot r\cdot\frac{-1}{q^2} = 0. \end{aligned} \tag{3.11}$$

Two-regular Case 6. Dihedral Extensions.

We assume that K/F is dihedral, that E/F, where $E = K^{\langle\sigma_\alpha\sigma_\beta\rangle}$, is an unramified extension of degree two, and that the square root i of -1 does not lie in F. We have the following congruences modulo $\mathfrak{p}$, the maximal ideal of O_K:

$$-1 \equiv c_1\sigma_\alpha\sigma_\beta(c_1) \equiv c_1^2 \equiv c.$$

This implies that $c_1 \equiv \pm i$. Conversely, if $c \in K^{\langle\sigma_\beta,\sigma_\alpha\sigma_\beta\sigma_\alpha\rangle}$ and $c \equiv -1 \mod \mathfrak{p}$, we obtain two solutions for c_1 in the biquadratic subfield of K. (The condition $c_1\sigma_\alpha(c_1) = 1$ requires $i \notin F$.)

We select a uniformizer $\pi' \in K$ such that $\pi'^4 \in E$, and a generator $\sigma = \sigma_\alpha\sigma_\beta$ of $\mathrm{Gal}\,(K/E)$, determining the sign of i by $\sigma(\pi') = i\pi'$. Replacing π' by an element of $\pi' O_E^\times$, we may assume that $\sigma_\beta(\pi') = \pi'$. Write $y_1 = \pi'^{\ell} y_1'$, with y_1' a unit. Equation 3.8 shows that ℓ is odd. Select a uniformizer π_1 in the ramified quadratic extension $K^{\langle\sigma_\beta,\sigma_\alpha\sigma_\beta\sigma_\alpha\rangle}$ that satisfies $\pi_1^2 \in F$.

Now we proceed as in Case 4. Set $x = (1 - c_1 t)/((\pi_1(1 + c_1 t))$. It is an F-coordinate, and $|x| \le 1 \Leftrightarrow |1 - c_1 t| < 1$. If $\ell = 1$, write

$$y_1 = \pi' i u_1 + \pi'^2 i u_2 + \pi'^3 i u_3,$$

with $u_1 \in O_E^\times$, $u_2, u_3 \in O_E$. The condition $\sigma_\beta(y_1) = -y_1$ of Equation 3.5 implies that $u_1 \in O_F^\times$, $u_2, u_3 \in O_F$. Following the argument of Case 4, we find that $Y_1 = Y' \cup Y''$, where

$$Y' = \{(u_1, u_2, x) : \ |u_1| = 1,\ |u_2| \le 1,\ |x| \le 1\},$$
$$Y'' = \{(u_1, u_2, u_3) : \ |u_1| = 1,\ |u_2| \le 1,\ |u_3| \le 1\},$$

$$|\omega'| = q^{1/4}\left|\frac{du_1\,du_2\,dx}{x^2}\right| \quad \text{on } Y',$$
$$|\omega'| = q^{-3/4}\,|du_1\,du_2\,du_3| \quad \text{on } Y''.$$

$$Z_1 = q^{1/4}\left(1 - \frac{1}{q}\right)\int_{\mathbb{P}^1}\left|\frac{dx}{x^2}\right|. \tag{3.12}$$

If $\ell = 3$, we write $y_1 = \pi'^3 i u_1 + \pi'^5 i u_2 + \pi'^6 i u_3$, with $u_1 \in O_F^\times$, $u_2, u_3 \in O_F$. We find

$$Z_3 = q^{-1/4}\left(1 - \frac{1}{q}\right)\int_{\mathbb{P}^1}\left|\frac{dx}{x^2}\right|. \tag{3.13}$$

By Lemma 2.2, these integrals are zero. This completes the proof of Theorem 3.3. □

4. The Cohomology of Cartan Subgroups

In the next section we will study some unstable two-regular germs in order to complete the matching of orbital integrals on Sp(4) with its endoscopic groups. The paper [H1] proved the matching for $GSp(4)$, but fell short of proving all that is needed for the unstable two-regular germs on Sp(4). We will rely on the formula of Lemma 3.1 to prove the required matching results.

In this section we review some well-known results about the cohomology of Cartan subgroups. For the unstable case, we need to study $H^1(\mathrm{Gal}\,(\bar F/F), T)$ where T is a Cartan subgroup over F of G. It is just as easy to state the results we need in greater generality. The Tate-Nakayama isomorphism, as described by Kottwitz [Ko], is a canonical isomorphism between the dual of $H^1(\mathrm{Gal}\,(\bar F/F), T)$ and the component group of the Galois invariant subgroup of the complex dual $\hat{T}$ to T, namely $\pi_0(\hat{T}^\Gamma)$, where, in this section only, Γ denotes the Galois group Gal $(\bar{F}/F)$.

The standard situation to which this is applied is the torus $T = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ (n factors), with character group $\{a_1\chi_1 + \cdots + a_n\chi_n : a_i \in \mathbb{Z}\} \simeq \mathbb{Z}^n$, where χ_i is the projection onto the ith factor of T. Let W be the Weyl group of B_n or C_n, the group of order $2^n n!$ of signed permutations. W is generated by a subgroup $\mathfrak{S}_n$, isomorphic to the symmetric group on n letters, and reflections ρ_i, $i = 1, \ldots, n$. The Weyl group acts on the character group of T by $\rho_i(\chi_j) = \pm\chi_j$ (negative only if $i = j$), and $\tau(\chi_j) = \chi_{\tau_j}$, for $\tau \in \mathfrak{S}_n$. If T is not split, then we assume that the Galois group acts on the character group through the Weyl group W, that is, we assume we are given a nontrivial map $\phi : \Gamma \to W$. The Weyl group W and hence Γ act naturally on the set I with n elements consisting of the unordered pairs $\{\chi_i, -\chi_i\}$.

We reduce to the case where Γ acts transitively on I. Let $I_1, \ldots, I_r$ be the distinct orbits. Then the image of Γ in the permutation group of I, under the natural map, lies in a subgroup isomorphic to $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_r}$, with $i_j = \#I_j$. Then T is a product $T_1 \times \cdots \times T_r$ over F and $\pi_0(\hat{T}^\Gamma) = \oplus\pi_0(\hat{T}_i^{\Gamma_i})$.

Lemma 4.1. *If Γ acts transitively on I, then $\pi_0(\hat{T}^\Gamma)$ is trivial unless there exists $\sigma \in \Gamma$ such that $\sigma(\chi_1) = -\chi_1$. In this case the component group has order two.*

Proof. By transitivity, we select $\sigma_j \in \Gamma$ and $\epsilon_j = \pm 1$ such that $\sigma_j(\chi_1) = \epsilon_j\chi_j$. Then $\hat{T}^\Gamma \subset \{(t, t^{\epsilon_1}, \ldots, t^{\epsilon_n})\} \simeq \mathbb{C}^\times$. This connected group is the full group of Galois-invariant elements unless an element σ exists with $\sigma(\chi_1) = -\chi_1$. Then $\sigma(t_1, \ldots, t_n) = (t_1^{-1}, \ldots)$, so $t_1 = \pm 1$. Thus $\hat{T}^\Gamma \subset \{\pm 1\} = \hat{T}^W \subset \hat{T}^\Gamma$, and $\hat{T}^\Gamma = \{\pm 1\}$. □

We remark that T is anisotropic if and only if the group of Galois invariants in the Lie algebra of $\hat{T}$ is trivial, or, equivalently, if the connected component of $\hat{T}^\Gamma$ is trivial. Thus the proof of the lemma shows that if Γ acts transitively on I, the cohomology is nontrivial exactly when T is anisotropic, and in this case it is cyclic of order two. In each of the following examples we assume, without loss of generality, that Γ acts transitively on I, and that T is anisotropic.

Example 1. $G = U_E(n)$, the unitary group split over the quadratic extension E. Let T_1 denote the intersection of T with $SU_E(n)$. The short exact sequence

$$1 \to T_1 \to T \to^{\det} U_E(1) \to 1$$

gives a long exact sequence in cohomology:

$$H^1(T) \to H^1(U_E(1)) \to H^2(T_1)$$

(we drop Gal $(\bar{F}/F)$ from the notation). Since T_1 is anisotropic, $H^2(T_1)$ is trivial [Shz], thus we have a surjective, hence isomorphic map between groups of order two $H^1(T) \twoheadrightarrow H^1(U_E(1))$. We conclude that the nontrivial class is detected by the determinant on $U_E(n)$.

Example 2. $G = \mathrm{SO}(2n+1)$ or $G = {}^E SO(2n)$ (the quasisplit group of type $\mathrm{SO}(2n)$ split over the quadratic extension E). We let T_{sc} denote the inverse image of T in the simply connected cover of G. We have an exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to T_{sc} \to T \to 1$$

and a long exact sequence in cohomology

$$H^1(T) \to H^2(\mathbb{Z}/2\mathbb{Z}) \to H^2(T_{sc}).$$

Since T_{sc} is anisotropic $H^2(T_{sc})$ is trivial. It is well-known that $H^2(\mathrm{Gal}(\bar{F}/F), \mathbb{Z}/2\mathbb{Z})$ is of order two. Thus we obtain an injective map from $H^1(T)$ to the Brauer group, from which we are able to detect the nontrivial class in the cohomology of T.

Example 3. Let G be any of the groups discussed in Examples 1 and 2. Set

$$\Gamma_1 = \{\sigma \in \Gamma : \sigma\chi_1 = \pm\chi_1\},$$
$$\Gamma_1^+ = \{\sigma \in \Gamma : \sigma\chi_1 = \chi_1\}.$$

Let E_1^+ be the fixed field in $\bar{F}$ of Γ_1^+, and E_1 the fixed field of Γ_1. We have $[E_1^+ : E_1] = 2$, and $[E_1 : F] = n$. We define a multiplicative homomorphism $E_1^\times \to H^1(\mathrm{Gal}(\bar{F}/F), T) = \{\pm 1\}$ whose kernel is the subgroup of norms from E_1^+. Let $U(1)$ be the form of $\mathbb{G}_m$ over E_1, whose E_1-points are elements of E_1^+ of norm 1. The torus T is isomorphic to the group $\mathrm{Res}_{E_1/F}\, U(1)$ over F obtained by restriction of scalars from $U(1)$. By Shapiro's lemma [Shz], the groups $H^1(\mathrm{Gal}(\bar{F}/F), \mathrm{Res}_{E_1/F}\, U(1))$ and $H^1(\mathrm{Gal}(\bar{F}/E_1), U(1))$ are isomorphic. The latter cohomology group is naturally identified with the quotient of $E_1^\times$ by the norms of E_1^+. In concrete terms, modifying a cocycle a_σ of $H^1(\mathrm{Gal}(\bar{F}/F), T)$ by a boundary, we may assume it satisfies $\chi_1(a_\sigma) = 1$, for all $\sigma \in \Gamma_1^+$. Then $\chi_1(a_\sigma)$ is an element of E_1 independent of $\sigma \in \Gamma_1 \setminus \Gamma_1^+$. Moreover, the cohomology class of a_σ determines $\chi_1(a_\sigma)$ modulo the norms of E_1^+.

5. Unstable Two-Regular Germs on Sp(4)

In this section we prove the transfer of κ-combinations of Shalika germs on Sp(4) to its endoscopic groups. The assumption that the residual characteristic of F is odd remains in effect. Let us survey what needs to be done. The elliptic endoscopic groups of Sp(4) are $SL(2) \times T_1$, where T_1 is a one-dimensional anisotropic torus, and the quasisplit forms of SO(4). The endoscopic groups $SL(2) \times T_1$ do not contain a two-regular unipotent conjugacy class, so in this case it must be shown that the relevant κ-combination of two-regular Shalika germs on Sp(4) vanishes. The split endoscopic group $H = \mathrm{SO}(4)$ has already been treated in [H1].

Suppose that H is a quasisplit form ${}^E SO(4)$ of SO(4), split over a nontrivial quadratic extension E/F. The adjoint group of H is isomorphic to the restriction of scalars $\mathrm{Res}_{E/F} PSL(2)$. Under this identification, the stable two-regular germ of H coincides with the subregular germ of $PSL(2)/E$. Let T be an elliptic torus in ${}^E SO(4)$. Write $T/\{\pm 1\} = \mathrm{Res}_{E/F} T_1$, for some elliptic torus $T_1 \subset PSL(2)/E$. Let Q be the twisted form of $\mathbb{P}^1 \times \mathbb{P}^1$, defined over E, attached to the torus T_1/E (see [LS1]). We conjugate an element of the Lie algebra of T by $H(\bar{F})$ to a diagonal parameter $\mathrm{diag}(t_1, t_2, -t_2, -t_1) \in \mathfrak{so}(4)$. Set $I_{ram}(q) := -(1+q)/q^{3/2}$, and $I_{unr}(q) := -2/q$.

Lemma 5.1. *The stable two-regular germ of T in ${}^E SO(4)$, evaluated at the element $\mathrm{diag}(t_1, t_2, -t_2, -t_1)$ of the Lie algebra, is*

$$|t_1^2 - t_2^2| \int_{Q(E)} |\omega_Q|,$$

where ω_Q is the differential form $dx_1 \wedge dx_2/(x_1 - x_2)^2$. More explicitly, if T_1 splits over a ramified extension of E, then

$$\int_{Q(E)} |\omega_Q| = I_{ram}(q_E),$$

and if T_1 splits over an unramified extension of E, then

$$\int_{Q(E)} |\omega_Q| = I_{unr}(q_E),$$

where q_E is the cardinality of the residue field of E.

Proof. The variety Q, the differential form $\omega_Q = dx_1 \wedge dx_2/(x_1 - x_2)^2$, and their relation to Shalika germs are treated thoroughly in [LS1]. The explicit formulas for the integrals appear in [SS,Lemma 2.5], after one accounts for different normalizations of measures. The integral over Q is the the constant denoted $-2(q+1)/(q^2\kappa_T)$ in [SS]. □

Fix a Cartan subgroup T in Sp(4) and attach to it a homomorphism $\phi : \mathrm{Gal}\,(\bar{F}/F) \to W$ as in previous sections. If $\mathrm{Im}(\phi) \subset \langle\sigma_\alpha, \sigma_\beta\sigma_\alpha\sigma_\beta\rangle$, then the endoscopic group is the split group SO(4) and the transfer of Shalika germs is carried out in [H1]. For any Cartan subgroup in Sp(4), the regular, subregular, and four-regular (trivial) unipotent classes are treated in [H1]. If T is not elliptic, then the theory of descent relates the Shalika germs to the Shalika germs on $SL(2)$. The two-regular germ is then zero, because $SL(2)$ does not contain any two-regular unipotent elements. These considerations show that the transfer of Shalika germs on Sp(4) will follow from the transfer of two-regular Shalika germs in the six cases enumerated in Lemma 3.2.

We look in turn at the κ-combinations of the two-regular Shalika germs for each of the six cases. For each character $\kappa : H^1(\mathrm{Gal}\,(\bar{F}/F), T) \to \mathbb{C}^\times$ we must integrate κ, evaluated at a given cocycle $t_\sigma \in H^1(\mathrm{Gal}\,(\bar{F}/F), T)$ depending on points of $X^*(F)$, over the manifold $X^*(F)$. By the results of Section 4, there are four possibilities for κ if T is a product of two one-dimensional anisotropic tori (Cases 1 and 2) and only one nontrivial possibility for κ if the elliptic torus is not a product (Cases 3,4,5, and 6).

The cocycle t_σ, by [H1,p240] is the cocycle of $\mathrm{Gal}\,(\bar{F}/F)$ obtained by $\phi : \mathrm{Gal}\,(\bar{F}/F) \to W$ from the cocycle of W:

$$\sigma_\alpha \to 1, \qquad \sigma_\beta \to (x(\delta)\cdot\ell)^{\beta^\vee},$$

where $\ell = (1 - tc_1)/(tc_1\Delta)$, $t = t_2/t_1$, $\Delta = t_2^2 - t_1^2$, and $x(\delta)$ is a constant in $F^\times$. (This expression for ℓ was obtained in the proof of Lemma 3.1.)

The formula of [H1,p240] has inadvertently dropped the constant $x(\delta)$. We are justified in dropping the factor here too: The cocycle $\sigma_\alpha \to 1$, $\sigma_\beta \mapsto x(\delta)$ does not depend on the point of the variety $X^*(F)$, but only contributes a sign to the normalization of an invariant measure on the two-regular unipotent class. Thus we assume $x(\delta) = 1$.

We set $z = 1 - tc_1$ and $z' = z/(1-z) = (1-tc_1)/(tc_1)$. If we let (u, v) represent the element $\mathrm{diag}(u, v, v^{-1}, u^{-1})$ in a Cartan subgroup T of Sp(4), then the cocycle t_σ of W, written out in full, is

$$\begin{aligned} 1,\ \sigma_\alpha &\mapsto (1,1),\\ \sigma_\beta,\ \sigma_\beta\sigma_\alpha &\mapsto (1, z'/\Delta),\\ \sigma_\alpha\sigma_\beta,\ \sigma_\alpha\sigma_\beta\sigma_\alpha &\mapsto (z/\Delta, 1),\\ \sigma_\beta\sigma_\alpha\sigma_\beta,\ \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta &\mapsto (z/\Delta, z'/\Delta). \end{aligned}$$

We evaluate $\kappa(t_\sigma)$ by Shapiro's lemma, as in Example 3 of Section 4. In Cases 1 and 2, the Cartan subgroup T is a product $T_1 \times T_2$, and $\kappa(t_\sigma)$ is

$$\eta(z\Delta),\quad \eta'(z'\Delta),\quad \text{or}\quad \eta(z\Delta)\eta'(z'\Delta).$$

The characters $\eta, \eta' : F^\times \to \mathbb{C}^\times$ of order two are attached to the quadratic extensions $K^{\langle\sigma_\beta\rangle}/F$ and $K^{\langle\sigma_\alpha\sigma_\beta\sigma_\alpha\rangle}/F$ respectively. The first two possibilities for $\kappa(t_\sigma)$ lead to the endoscopic groups $SL(2)\times T_1$ and $SL(2)\times T_2$, and the third possibility leads to ${}^E SO(4)$, where $E = K^{\langle\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\rangle}$. In Cases 3,4,5, and 6, the Cartan subgroup T is not a product and $\kappa(t_\sigma)$ is $\eta(z\Delta)$, where $\eta : E_1^\times \to \mathbb{C}^\times$ is the multiplicative character of $E_1 := K^{\langle\sigma_\beta,\sigma_\alpha\sigma_\beta\sigma_\alpha\rangle}$ attached to the quadratic extension $K^{\langle\sigma_\beta\rangle}/E_1$. The endoscopic group is the quasisplit form of SO(4) that splits over $K^{\langle\sigma_\alpha,\sigma_\beta\sigma_\alpha\sigma_\beta\rangle}$.

Since the cocycle depends on $X^*(F)$ only through the square root c_1 of the cross-ratio, we may easily insert the factor $\kappa(t_\sigma)$ into the formulas we have obtained in Section 3 to obtain expressions for the κ-germs. In light of Lemma 5.1, the following theorem establishes the transfer to the endoscopic groups of Sp(4). In the following theorem fix an elliptic Cartan subgroup T in Sp(4). Fix a character $\kappa : H^1(\mathrm{Gal}\,(\bar F/F), T) \to \mathbb{C}^\times$ and an associated endoscopic group H. If $H = {}^E SO(4)$, then we let Q be the variety attached to T and H as in Lemma 5.1. Let $\omega_Q = dx_1 \wedge dx_2/(x_1 - x_2)^2$ as above.

Theorem 5.2. *The integral* $\int_{X^*(F)} \kappa(t_\sigma)|\omega|$ *equals*
(1) zero, if H *is a product of* $SL(2)$ *and a one-dimensional torus,*
(2) $2|t_1^2 - t_2^2| \displaystyle\int_{Q(E)} |\omega_Q|$, *if* $H = {}^E SO(4)$ *and* E/F *is an unramified extension,*
(3) $\dfrac{2}{q}|t_1^2 - t_2^2| \displaystyle\int_{Q(E)} |\omega_Q|$, *if* $H = {}^E SO(4)$ *and* E/F *is ramified.*

Proof. We compute the integral $\int_{X^*(F)} \kappa(t_\sigma))|\omega|$ and compare the results with Lemma 5.1.

Part 1 of the theorem is elementary. Inserting the factor $\kappa(t_\sigma) = \eta'(z'\Delta)$ or $\eta(z\Delta)$ into Equation 3.4, and using $dz/z^2 = dz'/z'^2 = -tdc_1/(1 - tc_1)^2$, we find that the Shalika germ is a factor times

$$\int_{\mathbb{P}^1} \eta'(z'\Delta)\left|\frac{dz'}{z'^2}\right| \text{ or } \int_{\mathbb{P}^1} \eta(z\Delta)\left|\frac{dz}{z^2}\right|.$$

These integrals vanish by Lemma 2.2.

Before continuing further, we analyze the term $\eta(\Delta)$ for the various quadratic characters η appearing in $\kappa(t_\sigma)$. Recall that the character $\eta : E_1^\times \to \mathbb{C}^\times$ and the field E_1 depend on the case we are in. Let $\mathrm{sgn}(x)$, for $x \in F$, be 1 if x is a square in F, and -1 otherwise. In Cases 3,4, and 5, we see that $-(t_1 + t_2)^2$ is a norm, so $\eta(\Delta) = \eta(\dfrac{1-t}{1+t})$. In Case 3, η is an unramified character, and $(1-t)/(1+t)$ is a unit, so $\eta(\Delta) = 1$. In Case 4, t is congruent to a square root $i \in F$ of -1, and η is a ramified quadratic character, so $\eta(\Delta) = \eta(-i) = \mathrm{sgn}(2)$. In Case 5, $u := (1-t)/(1+t)$ satisfies $\sigma_\alpha\sigma_\beta(u)u = -1$, and η is a ramified character on E_1. Such a unit u will be a square in E_1 if and only if -1 is a square in F, so $\eta(\Delta) = \mathrm{sgn}(-1)$. In Case 6, $\Delta = -t_1^2(1 - t^2)$, where $t^2 \equiv -1 \mod \mathfrak{p}$, so $\eta(\Delta) = \eta(-2t_1^2)$. Also $-t_1^2$ is the norm of t_1, so $\eta(\Delta) = \mathrm{sgn}(2)$. In Cases 1 and 2, we must analyze $\eta''(\Delta) = \eta(\Delta)\eta'(\Delta)$, where η'' is the character attached to the quadratic extension $K^{\langle\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\rangle}/F$. In Case 1, η'' is unramified, and Δ has odd valuation, so $\eta''(\Delta) = -1$. In Case 2, η'' is ramified, and t_2^2 has even valuation, but is not a square in F, so $\eta''(\Delta) = \eta''(t_2^2) = -1$.

As we discuss Cases 1 through 6, we adopt the notation and conventions from the corresponding passages in Section 3. Turn to Cases 1 and 2 (biquadratic extensions). We assume the endoscopic group is a quasisplit form of SO(4). Inserting $\kappa(t_\sigma) = -\eta(z)\eta'(z/(1 - z))$

into Equation 3.4, we see that the germ is

$$2\left(\frac{1}{q}+\frac{1}{q^2}\right)|t_2^2-t_1^2|\int_{\mathbb{P}^1}\eta(z)\eta'(z/(1-z))\left|\frac{dz}{z^2}\right|.$$

In Case 1, SO(4) splits over an unramified extension, and the characters η, η' are distinct and ramified. In Case 2, SO(4) splits over a ramified extension, and the character η' is unramified, but η is ramified. The integral over $\mathbb{P}^1$ evaluates to $-(1+q^2)/(q^2+q)$ in Case 1 and to $-2/(q+1)$ in Case 2. The germ is then $2|t_2^2-t_1^2|I_{ram}(q^2)$ in Case 1 and $2|t_2^2-t_1^2|I_{unr}(q)/q$ in Case 2. Comparing these germs to Lemma 5.1, we see that Theorem 5.2 is verified in Cases 1 and 2.

In the remaining cases, we insert the factor $\kappa(t_\sigma)$ into the formula of Lemma 3.6 to obtain the expression

$$\begin{aligned}(5.3)\qquad &\frac{1}{|t_2^2-t_1^2|}\int_{X^*(F)}\kappa(t_\sigma)|\omega| \\ &\quad = \frac{2}{1-q}Z_0^\kappa+\frac{q^{1/4}(q+1)}{q(1-q)}Z_1^\kappa+\frac{q^{1/2}(q+1)}{q(1-q)}Z_2^\kappa+\frac{q^{3/4}(q+1)}{q(1-q)}Z_3^\kappa,\end{aligned}$$

where $Z_\ell^\kappa=\int_{Y_\ell}\kappa(t_\sigma)|\omega'|$. In Cases 3, 4, and 5, the extension E_1/F is equal to the quadratic extension E/F in K.

In Case 3, the character η is nontrivial and unramified. Referring to Equation 3.7, we see that $\kappa(t_\sigma)=\eta(z\Delta)=1$ at points for which $-1+\xi t\not\equiv 0$, and that when $-1+\xi t\equiv 0$, we may choose z and g_1 to be equal up to a unit so that $\eta(z\Delta)=\eta(g_1)$. Equations 3.7 and 5.3 then give the germ $2|t_1^2-t_2^2|Z_0^\kappa/(1-q)$, where

$$\begin{aligned}Z_0^\kappa &= \sum_{\substack{\xi\\ -1+\xi t\not\equiv 0}}\sum_{R(\xi)}\int|dg_1\,dg_2\,dg_3|+\sum_{\substack{\xi\\ -1+\xi t\equiv 0}}\sum_{R(\xi)}\int\eta(g_1)|g_1^{-2}dg_1\,dg_2\,dg_3| \\ &= q\cdot r\cdot\frac{1}{q^3}+1\cdot r\cdot\frac{1-q}{1+q}\cdot\frac{1}{q^2} \\ &= \frac{-2(1-q)}{q^2}=(1-q)I_{unr}(q^2).\end{aligned}$$

By Lemma 5.1 (with T_1 unramified and $q_E=q^2$), Theorem 5.2 holds in Case 3.

In Case 4, the character η is ramified. At points for which $|1+c_1t|<1$, we see that z is congruent to 2, and $\eta(z\Delta)=1$. At points for which $|1-c_1t|<1$, $\eta(z\Delta)$ is equal to $\eta(x)$, up to a constant, where x is the coordinate of Equations 3.9 and 3.10. The integral of the ramified character η over x is zero, so the terms associated with the set Y' in Equations 3.9 and 3.10 are zero. By Equation 5.3, this shows that the germ is the constant $|t_1^2-t_2^2|$ times

$$\begin{aligned}&\frac{q^{1/4}(q+1)}{q(1-q)}(Z_1^\kappa+q^{1/2}Z_3^\kappa) \\ &= \frac{q^{1/4}(q+1)}{q(1-q)}\left(\int_{Y''}q^{-3/4}|du_1\,du_2\,du_3|+q^{1/2}\int_{Y''}q^{-5/4}|du_1\,du_2\,du_3|\right) \\ &= \frac{-2(q+1)}{q\cdot q^{3/2}}=\frac{2}{q}I_{ram}(q)=\frac{2}{q}\int_{Q(E)}|\omega_Q|.\end{aligned}$$

Case 5 requires more effort. The character η on the quadratic extension $E^\times$ is ramified, but it restricts to a nontrivial unramified character $\eta_{E/F}$ on $F^\times$. By Equation 5.3 and the measure preceding Equation 3.11, the germ is $|t_1^2 - t_2^2|$ times

$$\frac{(q+1)}{q(1-q)} \int_{Y_2} \kappa(t_\sigma) \left| du_1\, du_2\, du_3 (1 - tc_1)^{-2} \right|.$$

In Equation 3.11, we analyze the terms $1 - tc_1 \not\equiv 0$ and $1 - tc_1 \equiv 0$ separately. Fix an element e in the residue field k_2 of E such that $\sigma(e) = -e$. If $1 - tc_1 \not\equiv 0$, then the reduction of tc_1 to the finite field has the form $(u+e)/(u-e)$ for some u in the residue field k of F. We have

$$\kappa(t_\sigma) = \eta(\Delta)\eta\left(1 - \left(\frac{u+e}{u-e}\right)\right) = \operatorname{sgn}(-1)\eta\left(\frac{-2e}{u-e}\right) = -\eta(u-e).$$

If the element $u - e$ is a square in k_2, then by taking norms, we find that u has the form $(a + e^2/a)/2$, for $a \in k^\times$; and conversely for any element of this form, we find that $u - e$ is a square in k_2. The map $a \mapsto a + e^2/a$ is two-to-one. Thus $\eta(u-e) = 1$ for $(q-1)/2$ values of $u \in k$, and $\eta(u-e) = -1$ for the remaining $(q+1)/2$ values. If $1 - tc_1 \equiv 0$, then we may take the F-coordinate g_1 to be $(-1 + c_1 t)/(e(1 + c_1 t))$ (lifting e to $O_E^\times$). Then $\eta(z\Delta) = -\eta_{E/F}(g_1)$. The unstable version of Equation 3.11 leads to the following formula for Z_2^κ:

$$\begin{aligned} q^{-1/2} \sum_{\substack{\xi \\ -1+t\xi \not\equiv 0}} \sum_{R(\xi)} \int \kappa(t_\sigma) |dg_1\, dg_2\, dg_3| + q^{-1/2} \sum_{\substack{\xi \\ -1+t\xi \equiv 0}} \sum_{R(\xi)} \int \eta_{E/F}(g_1) \left| \frac{dg_1}{g_1^2} dg_2\, dg_3 \right| \\ = q^{-1/2} \cdot \left(\frac{q+1}{2} - \frac{q-1}{2} \right) \cdot r \cdot \frac{1}{q^3} \quad - q^{-1/2} \cdot 1 \cdot r \cdot \left(\frac{1-q}{1+q} \right) \cdot \frac{1}{q^2} \\ = \frac{2(q-1)(1+q^2)}{q^{5/2}(1+q)} = 2 \frac{q(1-q)}{q^{1/2}(1+q)} I_{ram}(q^2). \end{aligned}$$

Inserting this into Formula 5.3 for the germ, and comparing to Lemma 5.1 (with T_1 ramified and $q_E = q^2$), we see that Theorem 5.2 holds in Case 5.

Case 6 is almost identical to Case 4. The character η is ramified, and the integral of $\eta(x)$ over the set Y' vanishes in Equations 3.12 and 3.13. The factor $\kappa(t_\sigma)$ equals 1 on Y''. The germ is then given by the same formula as in Case 4, and Theorem 5.2 holds in this final case. This concludes the proof of the transfer of germs on Sp(4) to its endoscopic groups. □

6. Appendix. A Table of Integrals

In this appendix, F is a nonarchimedean local field of characteristic zero, of odd residual characteristic. Let $\eta_i : F^\times \to \mathbb{C}^\times$ be the quadratic characters introduced in Section 2, associated with quadratic field extensions E_i/F, for $i = 1, 2, 3$. Set

$$L(a) = \int_F \log|1 - ax^2| \left| \frac{dx}{x^2} \right|, \quad \text{for } a \in F,$$

and

$$H_i(a) = \int_F \eta_i(1 - ax^2) \left| \frac{dx}{x^2} \right|, \quad \text{for } a \in F, \quad \text{and } i = 1, 2, 3.$$

(1) $$\int_F \theta(x) \left| \frac{dx}{x} \right| = 0, \quad \text{for any nontrivial quasicharacter } \theta : F^\times \to \mathbb{C}^\times.$$

(2) $$I_1(a) := \int_F \log|1 - ax| \left| \frac{dx}{x^2} \right| = |a| I_1(1) = 0.$$

(3) $$L(a) = I_1(\alpha) + I_1(-\alpha) = 0, \text{ if } a = \alpha^2 \in F^{\times 2}.$$

(4) $$L(a) = \frac{2|a|^{1/2}}{q-1}, \text{ if } \mathrm{val}(a) \in 2\mathbb{Z}, \text{ and } a \notin F^{\times 2}.$$

(5) $$L(a) = \frac{(q+1)|a\pi^{-1}|^{1/2}}{q(q-1)}, \text{ if } \mathrm{val}(a) \in 2\mathbb{Z}+1.$$

(6) $$H_i(a) = 0, \text{ if } \sqrt{a} \in E_i \text{ and } a \notin F^{\times 2}.$$

(7) $$H_1(a) = \frac{-4|a|^{1/2}}{(1+q)}, \text{ if } a \in F^{\times 2}.$$

(8) $$H_1(a) = \frac{-2|a\pi^{-1}|^{1/2}}{q}, \text{ if } \mathrm{val}(a) \in 2\mathbb{Z}+1.$$

(9) $$H_i(a) = \frac{-|a|^{1/2}(q+1)}{q}, \text{ if } \mathrm{val}(a) \in 2\mathbb{Z}, \text{ for } i = 2, 3.$$

(10) $$H_i(a) = \frac{-2|a\pi^{-1}|^{1/2}}{q}, \text{ if } \mathrm{val}(a) \in 2\mathbb{Z}+1 \text{ and } \sqrt{a} \notin E_i, \text{ for } i = 2, 3.$$

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