# **Orbital Integrals on $U(3)$**

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## 1. Statement of Theorem

Let $G = U_E(3)$ denote a unitary group in three variables over a nonarchimedean local field $F$ of characteristic zero. The group $G$ is quasi-split and splits over a quadratic extension $E$ of $F$. To be precise we assume that $G(F) = \{g \in GL(3,E) | {}^t\bar{g}Jg = J\}$ where $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. If $x$ is a matrix with coefficients in $E$, $\bar{x}$ denotes the matrix whose coefficients are conjugated by $Gal(E/F)$. The group of upper triangular matrices is a Borel subgroup of $G$ defined over $F$.

Let $H$ be the group $U_E(2) \times U_E(1)$ again quasi-split over $F$ and split over the quadratic field extension $E$ of $F$. Fix an embedding $\iota : H \to G$ defined over $F$ whose image is the subgroup

$$\left\{ \begin{pmatrix} * & & * \\ & * & \\ * & & * \end{pmatrix} \right\}.$$

$\iota$ is well defined up to an automorphism of $H$ over $F$.

Let $H'$ denote the image of $H$ in $G$. We identify Cartan subgroups $T$ in $H$ with Cartan subgroups $\iota(T)$ in $H'$ by this embedding. We drop $\iota$ from the notation when the context is clear.

There is an induced map on cohomology: $\iota_* : H^1(F,H) \to H^1(F,G)$. There is exactly one non-trivial class $a$ in $H^1(F,H)$ mapping to the trivial class in $H^1(F,G)$;

we let $a'$ denote the image of $a$ in $H^1(F,H')$. Let $e$ (resp. $e'$) denote the trivial class in $H^1(F,H)$ (resp. $H^1(F,H')$). Let $\overline{F}$ denote the algebraic closure of $F$. Fix $g_o \in G(\overline{F})$ such that $\sigma(g_o)g_o^{-1}$ for $\sigma \in Gal(\overline{F}/F)$ represents $a'$ in $H^1(F,H')$. Then $H'' =^{def} H'^{g_o}$ is a reductive group over $F$ in $G$ which is an inner form of $H'$.

Let $T \subset H'$ be a Cartan subgroup over $F$. Note that in general the quotient of the $F$-points $T(F)\backslash G(F)$ does not equal the $F$-points of the quotient $(T\backslash G)(F)$. The defining condition of the latter is $\sigma(Tg) = Tg$ or $\sigma(g)g^{-1} \in T(\overline{F})$ for all $\sigma \in Gal(\overline{F}/F)$. If $Tg \in (T\backslash G)(F)$ then the cocycle $\sigma(g)g^{-1}$, $\sigma \in Gal(\overline{F}/F)$, gives a well-defined class in $H^1(F,T)$ and hence in $H^1(F,H')$. Set $\kappa(\sigma(g)g^{-1}) = +1$ or $-1$ according as $\sigma(g)g^{-1}$ represents the trivial or non-trivial cocycle $e'$ or $a'$. It is a simple matter to see that $\kappa$ distinguishes between Cartan subgroups $T^g$ over $F$ conjugate by $G(F)$ into $H'$ from those $G(F)$-conjugate to one of $H''$.

Fix invariant forms $\omega_G$, $\omega_H$, $\omega_T$ of maximal degree on $G$, $H$, and $T$ for all Cartan subgroups $T$ of $H$. Assume they are defined over $F$. Identifying $T$ with a Cartan subgroup in $G$ by $\iota$ we obtain invariant forms $\omega_{T\backslash G}$ on $T\backslash G$ and $\omega_{T\backslash H}$ on $T\backslash H$ with associated measures $|\omega_{T\backslash G}|$ and $|\omega_{T\backslash H}|$.

Let $\Phi_G^{(T,\kappa)}(\gamma, f)$ for $f \in C_c^\infty(G)$ denote the integral $\int_{(T\backslash G)(F)} \kappa(\sigma(g)g^{-1}) f(g^{-1}\gamma g)|\omega_{T\backslash G}|$

for $\gamma$ regular in $T$. Similarly define $\Phi_H^{(T,st)}(\gamma, f^H) = \int_{(T\backslash H)(F)} f^H(h^{-1}\gamma h)|\omega_{T\backslash H}|$ for $\gamma$ regular in $T$ and $f^H \in C_c^\infty(H)$. $\Phi_G^{(T,\kappa)}(\gamma, f)$ and $\Phi_H^{(T,st)}(\gamma, f^H)$ are called $\kappa$-orbital integrals and stable orbital integrals respectively. Let $\gamma_1$, $\gamma_2$, $\gamma_3$ denote the eigenvalues of $\gamma$ chosen so that $\gamma_2$ is the projection of $\gamma$ from $H' = U_E(2) \times U_E(1)$ to $U_E(1)$. We consider the following functions

$$D_H(\gamma) = \frac{|\gamma_1 - \gamma_3|}{|\gamma_1\gamma_3|^{1/2}}$$

$$\Delta_G^*(\gamma) = \Delta(\gamma)D_H(\gamma)$$

for $\gamma$ having regular image in $G$ ($G$-regular). Here $\Delta(\gamma)$, or rather $\Delta(\gamma, \iota(\gamma))$, is the transfer factor of Langlands and Shelstad [LS2]. It is defined in the next section.

The purpose of this paper is to sketch a proof of the following theorem of Langlands and Shelstad. [L2], [LS3].

**Theorem (Langlands-Shelstad) 1.1.** *For every $f \in C_c^\infty(G)$ there exists a function $f^H \in C_c^\infty(H)$ such that for all $T \subseteq H$ and $\gamma$ with $\iota(\gamma)$ regular in $G$*

$$D_H(\gamma)\Phi_H^{(T,st)}(\gamma, f^H) = \Delta_G^*(\iota(\gamma))\Phi_G^{(T,\kappa)}(\iota(\gamma), f)$$

The outline of the proof is as follows. In the next section the transfer factor of Langlands and Shelstad is defined, and a few of its elementary properties are described. Section 4 discusses descent which reduces the theorem to elements $\gamma$ in an arbitrarily small neighborhood of the identity. Near the identity element there is an expansion, called the Shalika germ expansion, for orbital integrals. We will see that our $\kappa$-orbital integrals possess a three term expansion. To prove the theorem we must match each term with a corresponding term in the expansion of stable orbital integrals on $H$. Sections 7 and 8 describe a general method, Igusa theory, of obtaining formulas for Shalika germs. Section 9 matches the first term of the expansion, the regular germ, with the corresponding term on the endoscopic group. Sections 10 and 11 show that one of the remaining two terms of the Shalika germ expansion vanishes, while the other matches the remaining term on $H$. Section 12 relates the normalization of measures to the identity of the Hecke algebra. Section 13 shows how to extend the results to other endoscopic groups.

## 2. Transfer Factors

The paper [LS2] of Langlands and Shelstad defines a factor $\Delta(\gamma) = \Delta(\gamma, \iota(\gamma))$ associated to strongly $G$-regular semisimple elements $\gamma$ in $H'$. This section gives an explicit formula for that factor. The factor $\Delta(\gamma)$ is defined as a product of 5 factors

$$\Delta_I, \quad \Delta_{II}, \quad \Delta_{III_1}, \quad \Delta_{III_2}, \quad \Delta_{IV}$$

The product of the factors is canonically defined. The individual factors rely on several choices. We remark that the factor $\Delta(\gamma)$ is actually a function $\Delta(\gamma_H, \gamma_G)$, $\gamma_H \in H(F)$, $\gamma_G \in G(F)$ and that $\Delta(\gamma_H, \gamma_G)$ itself is defined through a factor depending on four elements ($\Delta(\gamma_H, \gamma_G, \bar{\gamma}_H, \bar{\gamma}_G)$, $\gamma_H, \bar{\gamma}_H \in H(F)$, $\gamma_G, \bar{\gamma}_G \in G(F)$). A pair $\bar{\gamma}_H, \bar{\gamma}_G$ is fixed, $\Delta(\bar{\gamma}_H, \bar{\gamma}_G)$ is specified arbitrarily and $\Delta(\gamma_H, \gamma_G)$ is defined as

$$\Delta(\bar{\gamma}_H, \bar{\gamma}_G)\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G).$$

What we write as $\Delta(\gamma)$ is the factor $\Delta_0(\gamma, \iota(\gamma))$ of [LS2] for an appropriate choice of $F$-splitting.

Let $\alpha = \gamma_1 {\gamma_2}^{-1}$, $\beta = \gamma_2 {\gamma_3}^{-1}$ denote the positive simple roots of the diagonally embedded Cartan subgroup $\mathbf{T}$ in the upper triangular Borel subgroup $\mathbf{B}$. At times we will write the root $\alpha\beta$ additively as $\alpha + \beta$. Let $X^*(\mathbf{T})$ denote the character group of $\mathbf{T}$. The Galois group of $E/F$ and the Weyl group $\Omega$ act on $X^*(\mathbf{T})$. These actions may be combined to give an action of the semidirect product $\Omega \rtimes Gal(E/F)$ on $X^*(\mathbf{T})$. We let $\omega$ denote the longest element $\omega = \sigma_\alpha \sigma_\beta \sigma_\alpha$ in $\Omega$ ($\sigma_\alpha, \sigma_\beta$ simple reflections), and let $\tau$ denote the nontrivial element in $Gal(E/F)$. Let $\Theta$ denote the subgroup $\{1, \tau, \omega, \tau\omega\}$ of $\Omega \rtimes Gal(E/F)$. It is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.

If $T$ is a Cartan subgroup of $H'$ over $F$ split by a Galois extension $K$ of $F$ then by conjugating $T$ to $\mathbf{T}$ we identify the roots of $T$ with $\alpha, \beta, \alpha\beta, \alpha^{-1}, \beta^{-1}, \alpha^{-1}\beta^{-1}$ and we identify $X^*(T)$ with $X^*(\mathbf{T})$. Transporting the action of $Gal(K/F)$ on $X^*(T)$ to $X^*(\mathbf{T})$ one finds that there is a unique map

$$Gal(K/F) \mapsto^{\phi_T} \Theta$$

with image $\Gamma_T$ respecting the transported action of $Gal(K/F)$ and $\Theta$ on $X^*(\mathbf{T})$. Note also that we have a commutative diagram

$$\begin{array}{ccc} Gal(K/F) \to^{\phi_T} & \Theta \subseteq & \Omega \rtimes Gal(E/F) \\ \searrow & & \swarrow \\ & Gal(E/F) & \end{array}$$

We must distinguish three possibilities for $T$. These are the only $T$ that are stably conjugate to a Cartan subgroup of $H'$ or $H''$:

| | | | |
|---|---|---|---|
| (i) | $K = E$ | $\Gamma_T = \{1, \tau\}$ | $Gal(E/F) = \{1, \sigma_\tau\}$ |
| (ii) | $K = E$ | $\Gamma_T = \{1, \omega\tau\}$ | $Gal(E/F) = \{1, \sigma_{\omega\tau}\}$ |
| (iii) | $[K : F] = 4$ | $\Gamma_T = \{1, \tau, \omega, \omega\tau\}$ | $Gal(K/F) = \{1, \sigma_\tau, \sigma_\omega, \sigma_{\omega\tau}\}$ |

In case $(iii)$, $E$ is the field fixed by $\sigma_\omega$, $E_\tau$ is the field fixed by $\sigma_\tau$, and $E_{\omega\tau}$ is the field fixed by $\sigma_{\omega\tau}$.

A certain selection of constants in $\overline{F}^\times$, known as $a$-data and a certain selection of characters on extensions of $F^\times$ known as $\chi$-data, are required for the definition of $\Delta_I$, $\Delta_{II}$, $\Delta_{III_1}$, $\Delta_{III_2}$. The following table indicates choices of $a$-data and $\chi$-data and the corresponding definitions of factors. The reader may consult [LS2] for details. We also select the element $h$ of the construction to lie in the derived group of $H'$. If $E''/E'$ is a quadratic extension of fields let $\eta_{E''/E'} : E'^{\times} \to \{\pm 1\}$ denote the character of $E'^{\times}$ associated to this extension by local class field theory.

| | (i) | (ii) |
|---|---|---|
| $a$-data | $a_\alpha = a_\beta = a_{\alpha+\beta} = 1$<br>$a_{-\alpha} = a_{-\beta} = a_{-\alpha-\beta} = -1$ | $a_\alpha = -a_\beta = a_{\alpha+\beta} \in E^\times$<br>$\bar{a}_\alpha = -a_\alpha$<br>$a_{-\alpha} = -a_{-\beta} = a_{-\alpha-\beta} = -a_\alpha$ |
| $\chi$-data | all trivial | $\chi_\alpha = \chi_\beta$ characters on $E^\times$<br>extending<br>$\eta_{E/F} : F^\times \to \{\pm 1\}$ |
| $\Delta_I(\gamma)$ | 1 | 1 |
| $\Delta_{II}(\gamma)$ | 1 | $\chi_\alpha((\gamma_1{\gamma_2}^{-1} - 1)(\gamma_2{\gamma_3}^{-1} - 1))$ |
| $\Delta_{III_1}(\gamma)$ | 1 | 1 |
| $\Delta_{III_2}(\gamma)$ | $\theta_1(\gamma)$ | $\theta_2(\gamma)$ |
| $\Delta_{IV}(\gamma)$ | $D_{G/H}(\gamma) = \lvert(\gamma_1{\gamma_2}^{-1} - 1)({\gamma_1}^{-1}\gamma_2 - 1)(\gamma_2{\gamma_3}^{-1} - 1)({\gamma_2}^{-1}\gamma$ | |
| $\Delta(\gamma)$ | $\theta_1(\gamma) D_{G/H}(\gamma)$ | $\theta_2(\gamma)\chi_\alpha((\gamma_1{\gamma_2}^{-1} - 1)(\gamma_2{\gamma_3}^{-1} - 1)) D_{G/H}(\gamma)$ |

| | (ii) | (iii) |
|---|---|---|
| $= 1$<br>$= -1$ | $a_\alpha = -a_\beta = a_{\alpha+\beta} \in E^\times$<br>$\bar{a}_\alpha = -a_\alpha$<br>$a_{-\alpha} = -a_{-\beta} = a_{-\alpha-\beta} = -a_\alpha$ | $a_\alpha \in E^\times,\ \bar{a}_\alpha = -a_\alpha$<br>$a_{\alpha+\beta} \in E_\tau,\ \sigma_\omega(a_{\alpha+\beta}) = -a_{\alpha+\beta}$<br>$a_{-*} = -a_*,\ * = \alpha, \beta, \alpha+\beta$<br>$a_\beta = -a_\alpha$ |
| | $\chi_\alpha = \chi_\beta$ characters on $E^\times$<br>extending<br>$\eta_{E/F} : F^\times \to \{\pm 1\}$ | $\chi_\alpha : K^\times \to \mathbf{C}^\times$<br>extending<br>$\eta_{K/E_{\omega\tau}} : E_{\omega\tau}^\times \to \{\pm 1\}$ |
| | 1 | 1 |
| | $\chi_\alpha((\gamma_1{\gamma_2}^{-1} - 1)(\gamma_2{\gamma_3}^{-1} - 1))$ | $\chi_\alpha\left(\dfrac{\gamma_1{\gamma_2}^{-1} - 1}{a_\alpha}\right)$ |
| | 1 | 1 |
| | $\theta_2(\gamma)$ | $\theta_3(\gamma)$ |
| $D_{G/H}(\gamma) = \lvert(\gamma_1{\gamma_2}^{-1} - 1)({\gamma_1}^{-1}\gamma_2 - 1)(\gamma_2{\gamma_3}^{-1} - 1)({\gamma_2}^{-1}\gamma_3 - 1)\rvert^{1/2}$ | | |
| | $\theta_2(\gamma)\chi_\alpha((\gamma_1{\gamma_2}^{-1} - 1)(\gamma_2{\gamma_3}^{-1} - 1))D_{G/H}(\gamma)$ | $\theta_3(\gamma)\chi_\alpha\left(\dfrac{\gamma_1{\gamma_2}^{-1} - 1}{a_\alpha}\right)D_{G/H}(\gamma)$ |

$\theta_1$, $\theta_2$, $\theta_3$ are continuous characters on the $F$-points of the Cartan subgroups $T = C_G(\gamma)$. We will see in section 4 that it is not necessary to compute $\theta_i$ to prove theorem 1.1. since for $\gamma$ sufficiently close to 1 we have $\theta_i(\gamma) = 1$.

Here is a brief description of $\theta_i$. To fix the characters $\theta_1, \theta_2, \theta_3$ it is necessary to fix supplementary data, endoscopic data $(H, \mathcal{H}, s, \xi)$ attached to the endoscopic group $H$. Let $W_F$ denote the Weil group of $F$ and $\hat{G}$, $\hat{H}$, $\hat{T}$ the connected complex duals of $G$, $H$, and $T$. Examples: $\widehat{U(n)} = GL(n, \mathbf{C})$, $\widehat{Sp(2n)} = SO(2n+1, \mathbf{C})$,

$\widehat{SL(n)} = PSL(n, \mathbf{C})$; the Weil group $W_{E/F}$, at a finite level, is a nonsplit extension

$$1 \to E^\times \to W_{E/F} \to Gal(E/F) \to 1$$

$\mathcal{H}$ and ${}^L G$ are defined as semidirect products

$$1 \to \hat{H} \to \mathcal{H} \to W_F \to 1$$

$$1 \to \hat{G} \to {}^L G \to W_F \to 1$$

satisfying certain properties. The element $s$ is semisimple in $\hat{G}$. Finally, $\xi$ is an embedding $\xi : \mathcal{H} \to {}^L G$ which restricts to an isomorphism of $\hat{H}$ and $C_{\hat{G}}(s)^0$. This data is subject to a list of conditions [L1,LS2] and a notion of equivalence. The $\chi$-data and $\xi$ determine an element $a \in H^1(W_F, \hat{T})$. This cohomology group pairs with $T(F)$ as in [B], and determines a character $\theta_i$ of $T(F)$.

It is also necessary to have a limit formula for $\Delta(\gamma)$. Set $\gamma = \exp(\lambda X)$, $\lambda \in F^\times$. Let $\alpha(X)$, $\beta(X)$ be the simple roots evaluated on the element $X$ of the Lie algebra.

**Lemma 2.1.**

$$\Delta_0(X) =^{def} \lim_{\lambda \to 0} \frac{\Delta(\gamma)}{D_{G/H}(\gamma)} = \eta_{E/F}(\alpha(X)\beta(X))$$

**Proof.** This is immediate from the definitions of $\Delta(\gamma)$ in cases (i) and (ii). In case (iii)

$$\Delta_0(X) = \lim \eta_{K/E_{\omega\tau}}(\frac{\lambda\alpha(X)}{a_\alpha})$$

with $\frac{\lambda\alpha(X)}{a_\alpha} \in E_{\omega\tau}$. By local class field theory this is equal to

$$\eta_{E/F}\left(\frac{\lambda\alpha(X)}{a_\alpha}\tau\left(\frac{\lambda\alpha(X)}{a_\alpha}\right)\right) = \eta_{E/F}(\alpha(X)\beta(X)).$$

Another limit of interest is

**Lemma 2.2.** *For $\gamma$ sufficiently small*

$$\lim_{\gamma_1 \to \gamma_3} \frac{\Delta(\gamma)}{D_{G/H}(\gamma)} = 1$$

**Proof.** We may select $\gamma$ small so that $\theta_i(\gamma) = 1$. The lemma is immediate from the definitions in cases (i),(ii). For case (iii) and $\gamma$ sufficiently small the previous lemma may be used to approximate $\Delta(\gamma)$ by $\eta_{E/F}(\alpha(X)\beta(X))$, noting that $\alpha(X) \in E$ when $\gamma_1 = \gamma_3$ so that $\alpha(X)\beta(X)$ is the norm of $\alpha(X)$.

Although this paper deals with $F$ a $p$-adic field, [LS3] also defines transfer factors in the case of $F$ archimedean. It is enough to consider the case $F = \mathbf{R}$ by restricting scalars for complex groups. Shelstad in [S] defines a transfer factor

$\Delta^{(\mathbf{R})}(\gamma_H, \gamma_G)$ for real groups and proves the matching of orbital integrals. In [LS4] Langlands and Shelstad show that there is a constant $c$ such that

$$c\Delta^{(\mathbf{R})}(\gamma_H, \gamma_G) = \Delta(\gamma_H, \gamma_G)$$

for all $G$-regular $\gamma_H$ in $H(\mathbf{R})$. The matching of orbital integrals for $\Delta(\gamma_H, \gamma_G)$ then follows by this work of Shelstad.

There is also a global product formula relating the transfer factors at all places of a number field $k$ with the ring of adeles $\mathbf{A}$. We take $H$ and $G$ to be over $k$. We say that $\gamma_H \in H(k)$ is an adelic image of $\gamma_G \in G(\mathbf{A})$ if for every place $v$, $\gamma_H$ is an image of $\gamma_{G,v}$ in $G(F_v)$.

Using global Tate-Nakayama duality, Langlands and Shelstad define a global factor $\Delta_{\mathbf{A}}(\gamma_H, \gamma_G) \in \mathbf{C}^\times$, $\gamma_H \in H(k)$, $\gamma_G \in G(\mathbf{A})$, which coincides with the term $\kappa(\epsilon(D))$ of [L1]. It depends only on the stable conjugacy class of $\gamma_H$, and the $G(\mathbf{A})$-conjugacy class of $\gamma_G$. We insert the superscript $(v)$ on local factors. They prove the product formula:

$$\Delta^{(v)}(\gamma_H, \gamma_{G,v}) = 1 \text{ for almost all } v$$

$$\prod_v \Delta^{(v)}(\gamma_H, \gamma_{G,v}) = \Delta_{\mathbf{A}}(\gamma_H, \gamma_G).$$

## 3. Shalika Germs

As above let $G$ be a reductive group over a $p$-adic field of characteristic zero, and let $T$ be a Cartan subgroup over $F$. For every unipotent orbit $O$ in $G(F)$ fix an invariant measure $\mu_O$ on $O$. Shalika [Sh] has shown that there exist functions $\Gamma_O^T(\gamma)$ called germs defined on the regular elements of $T(F)$ for all unipotent classes $O$ in $G(F)$ such that for every $f \in C_c^\infty(G)$ there is a neighborhood $V_f$ of the identity element in $T(F)$ in which the expansion

$$\int_{T(F)\backslash G(F)} f(g^{-1}\gamma g)|\omega_{T\backslash G}| = \sum_O \mu_O(f)\Gamma_O^T(\gamma)$$

holds for $\gamma$ regular in $V_f$.

Similarly for $\kappa$ and stable orbital integrals there are expansions, called Shalika germ expansions, in sufficiently small neighborhoods of the identity:

$$\Delta_G^*(\gamma)\Phi_G^{(T,\kappa)}(\gamma, f) = \sum_O \mu_O(f)\Gamma_O^{(T,\kappa)}(\gamma)$$

$$D_H(\gamma)\Phi_H^{(T,st)}(\gamma, f^H) = \sum_{O_H} \mu_{O_H}(f^H)\Gamma_{O_H}^{(T,st)}(\gamma)$$

where $O_H$ runs over unipotent classes of $H(F)$. The most difficult part of theorem 1.1 is to obtain an expression for the Shalika germs $\Gamma_O^{(T,\kappa)}, \Gamma_{O_H}^{(T,st)}$.

## 4. Descent

In this section $G$ is any connected reductive group over $F$ and $H$ is an endoscopic group of $G$. Orbital integrals display complicated behavior near the identity element. More generally, near any fixed element $\gamma_0$, an orbital integral behaves as an appropriate chosen orbital integral on the centralizer $C_G(\gamma_0)$. This section gives a precise form to this assertion. This section also indicates how Langlands and Shelstad have used this descent property to reduce the proof of theorem 1.1 to statements about Shalika germs near the identity element.

We recall of Rogawski's form of Harish-Chandra descent [R1]. Let $M$ be a connected reductive subgroup over $F$ of $G$ containing a Cartan subgroup of $G$. Let $\gamma_0 \in M(F)$ be such that $C_G(\gamma_0) \subseteq M$. Let $T_1, \ldots, T_j$ be the Cartan subgroups over $F$ of $M$ containing $\gamma_0$ up to conjugacy by $M(F)$. Let $\Omega$ be a compact open set in $G(F)$. Then there exists [R1 lemma1] a compact open set $K$ of $G(F)$ depending on $M$ and $\Omega$ and there exist neighborhoods $V_i$ of $\gamma_0$ in $T_i(F)$ such that $\{g \in G(F) | g^{-1} V_i g \cap \Omega \neq \emptyset\} \subseteq M(F)K$ for $i = 1, \ldots, j$. Fix invariant forms $\omega_G$, $\omega_M$,

$\omega_{T_i}$ on $G$, $M$, and $T_i$. We obtain compatible forms $\omega_{T_i\backslash G}$, $\omega_{T_i\backslash M}$, $\omega_G$ and $\omega_M$ and corresponding measures on $T_i(F)\backslash G(F)$, $T_i(F)\backslash M(F)$, $G(F)$, $M(F)$. Further choosing $\alpha \in C_c^\infty(G)$ so that

$$\int_{M(F)} \alpha(mg)|\omega_M| = \begin{cases} 1 & g \in M(F)K \\ 0 & \text{otherwise,} \end{cases}$$

then for $f \in C_c^\infty(\Omega)$ and $\gamma_i \in V_i \cap T_i^o(F)$ ($T_i^o(F)$ = the regular elements of $T_i(F)$) we have

$$\int_{T_i(F)\backslash G(F)} f(g^{-1}\gamma_i g)|\omega_{T_i\backslash G}| = \int_{T_i(F)\backslash M(F)} \lambda(f)(m^{-1}\gamma_i m)|\omega_{T_i\backslash M}|,$$

$i = 1, \dots, j$, where $\lambda : C_c^\infty(\Omega) \to C_c^\infty(M)$ is the linear functional

$$\lambda(f)(m) = \int_{G(F)} \alpha(g) f(g^{-1}mg)|\omega_G|.$$

We note that $\lambda$ depends on $\Omega, M, \gamma_0, K, \alpha$.

The rest of this section mentions a few results of [LS3]. Two points encumber the notation of this section. First, for an arbitrary reductive group $G$, the matching of orbital integrals is not between $G$ and an endoscopic group $H$, but between $G$ and $\tilde{H}$ a central extension of $H$. The transfer factor is then a function $\Delta(\tilde{\gamma}_H, \gamma_G)$ pairing certain elements of $\tilde{H}(F)$ and $G(F)$. Near the identity element $\Delta(\tilde{\gamma}_H, \gamma_G)$

descends to a factor $\Delta_{loc}(\gamma_H, \gamma_G)$ on $H$ and $G$. The functions $f^{\tilde{H}}$ in $C_c^\infty(\tilde{H})$ are to be functions $C_c^\infty(\tilde{H}, \tilde{\lambda})$ transforming by a specified character $\tilde{\lambda}$ on the center of $\tilde{H}(F)$. Second we must use the relative factor $\Delta(\tilde{\gamma}_H, \gamma_G; \bar{\tilde{\gamma}}_H, \bar{\gamma}_G) = \dfrac{\Delta(\tilde{\gamma}_H, \gamma_G)}{\Delta(\bar{\tilde{\gamma}}_H, \bar{\gamma}_G)}$ mentioned at the beginning of section 2.

Start with a (possibly singular) semisimple element $\epsilon_H$ in $H(F)$ an *image* of $\epsilon_G$ in $G(F)$. (For the general definition of image see [LS3]. For $H = U_E(2) \times U_E(1)$, we say that $\epsilon_H$ is an image of $\epsilon_G$ if it is conjugate to $\iota(\epsilon_H)$ or $\iota(\epsilon_H)^{g_0}$.) The elements $\epsilon_H$ and $\epsilon_G$ have connected centralizers $H_{\epsilon_H}$ and $G_{\epsilon_G}$ in $H$ and $G$. We assume that $\epsilon_H$ is such that $H_{\epsilon_H}$ is quasisplit. From the endoscopic data $(H, \mathcal{H}, s, \xi)$ for $G$ they define endoscopic data $(H_{\epsilon_H}, \mathcal{H}_\epsilon, s, \xi_\epsilon)$ for $G_{\epsilon_G}$. Associated to this endoscopic data is a transfer factor denoted $\Delta_\epsilon$.

We say that $(G, H)$ admits *local* $\Delta$-transfer at the identity if for any $f \in C_c^\infty(G)$ there exists $f^H \in C_c^\infty(H)$ such that

$$\Phi^{st}(\gamma, f^H) = \sum_{\gamma_G} \Delta_{loc}(\gamma_H, \gamma_G) \Phi(\gamma_G, f)$$

for strongly $G$-regular $\gamma_H$ near 1 in $H(F)$.

We say that $(G, H)$ admits $\Delta$-transfer if for any $f \in C_c^\infty(G)$ there exists $f^{\tilde{H}} \in$

$C_c^\infty(\tilde{H}, \tilde{\lambda})$ such that

$$\Phi^{st}(\tilde{\gamma}_H, f^{\tilde{H}}) = \sum_{\gamma_G} \Delta(\tilde{\gamma}_H, \gamma_G)\Phi(\gamma_G, f) \qquad (*)$$

for strongly $G$-regular $\tilde{\gamma}_H$ in $\tilde{H}(F)$.

**Remark.** The sum runs over conjugacy classes of strongly regular elements in $G(F)$. All but finitely many terms are zero. This sum effectively replaces an integral over $T(F)\backslash G(F)$ by an integral over $(T\backslash G)(F)$. For $G = U_E(3)$ the right and left hand sides of (*) coincide, up to the factor $D_H(\gamma)$, with the right and left hand sides of the equation of theorem 1.1.

The main result of [LS3] is

**Theorem 4.1.** *Suppose that all pairs $(G_{\epsilon_G}, H_{\epsilon_H})$ admit local $\Delta_\epsilon$-transfer at the identity. Then $(G, H)$ admits $\Delta$-transfer.*

Several remarks are in order.

1. Three simplifications occur in local $\Delta$-transfer. First $\Delta_{loc}$ is a function on $H(F)$ and $G(F)$ whereas $\Delta$ in general is a function on the central extension $\tilde{H}$ and $G$. Second the term $\Delta_{III_2}$, the term of the greatest complexity in the definition of the transfer factor, is identically 1 near the identity. Third the Shalika germ

expansion holds near the identity. Thus theorem 4.1 allows us to use the simplified transfer factor $\Delta_{loc}$, ignore the central extension $\tilde{H}$ and to match terms of the Shalika germ expansion term by term in proving theorem 1.1. This is carried out in sections 6-11.

2. The proof of the theorem 4.1 is elementary once it is known that the transfer factors satisfy

$$\lim_{\substack{\tilde{\gamma}_H, \bar{\tilde{\gamma}}_H \to \tilde{\epsilon}_H \\ \gamma_G, \bar{\gamma}_G \to \epsilon_G}} \frac{\Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G)}{\Delta_\epsilon(\tilde{\gamma}_H, \gamma_G; \bar{\tilde{\gamma}}_H, \gamma_G)} = 1 \qquad (**)$$

However, the proof of (**) is extremely technical and occupies the bulk of [LS3].

3. In the case of $G = U_E(3)$, one can check that if $G_{\epsilon_G}$ is not conjugate to a subgroup of $H'$ or $H''$ and if $\epsilon_G$ is conjugate to an element of $H'(F)$ or $H''(F)$, then $\epsilon_G$ is central in $G$. In the situation where $G_{\epsilon_G}$ is conjugate to a subgroup of $H'$ or $H''$, then $H_{\epsilon_H}$ is the quasisplit inner form of $G_{\epsilon_G}$. Further, $G_{\epsilon_G}$ if not a Cartan subgroup must have a form of $SL(2)$ as derived group. This situation, that of SL(2), has been extensively studied in [LS1]. There it is shown that local $\Delta_\epsilon$-matching holds for such $(G_{\epsilon_G}, H_{\epsilon_H})$.

Thus in the case of $G = U_E(3)$ the only remaining case to be studied is that of $\epsilon_G$ central in $G$, so that we may assume $G_{\epsilon_G} = G, H_{\epsilon_H} = H$. Thus theorem 4.1

becomes : suppose that $(G,H)$ admits *local* $\Delta$-transfer at the identity, then $(G,H)$ has $\Delta$-transfer. We can restate theorem 1.1 as

**Theorem 4.2.** *$(G,H)$ admits local $\Delta$-transfer at the identity.*

4. Note that lemma 2.2 is a special case of (**), for if $\epsilon_G = \iota(\gamma)$ with $\gamma = \epsilon_H$, $\gamma_1 = \gamma_3$, then $G_{\epsilon_G} = H'_{\epsilon_H}$, $\Delta_\epsilon(\tilde{\gamma}_H, \gamma_G, \tilde{\bar{\gamma}}_H, \gamma_G)$ is seen to be 1, and by 2.2 (near the identity)

$$\lim_{\substack{\tilde{\gamma}_H, \tilde{\bar{\gamma}}_H \to \tilde{\epsilon}_H \\ \gamma_G, \bar{\gamma}_G \to \epsilon_G}} \Delta(\gamma_H, \gamma_G; \bar{\gamma}_H, \bar{\gamma}_G) = 1.$$

Thus (**) holds in this special case.

## 5. Unipotent Classes

It is known that the germ $\Gamma_1^T$ associated to the identity element $1 \in G(F)$ is a constant independent of the elliptic Cartan subgroup $T$ and is zero if $T$ is not elliptic [Ho,HC,R2]. A simple general argument using this fact shows that $\Gamma_1^{(T,\kappa)} = 0$ when $\kappa$ is non-trivial. The argument is given in [LS3]. We turn to the other unipotent classes.

Over $\overline{F}$ the group $G(\overline{F})$ has 3 unipotent conjugacy classes corresponding to the various partitions of $3:\ 3,\ 21,\ 1^3$. These are the regular unipotent class, subregular unipotent class, and the identity $\{1\}$. Over $F$, the $\overline{F}$ regular class remains a single unipotent class but the subregular class breaks into 2 classes. This calculation can be easily seen from conjugating the regular and subregular elements to

$$\begin{pmatrix} 1 & 1 & -\frac{1}{2} \\ & 1 & -1 \\ & & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & ux \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

where $u \in F^\times$ and $x \in E^\times$, $\bar{x} = -x$. Conjugation by an element $diag(a, b, c) \in U_E(3, F)$ only changes $u \in F$ by a norm from $E$.

Fix an invariant form $\omega_{sub}$ on the stable subregular class. Then set $\mu^+_{sub} = |\omega_{sub}|_{O_o} + |\omega_{sub}|_{O_1}$ and $\mu^-_{sub} = |\omega_{sub}|_{O_o} - |\omega_{sub}|_{O_1}$ where $O_o$ and $O_1$ are the two subregular classes over $F$. We shall write the unipotent measures as $\mu_{reg}$, $\mu^\pm_{sub}$, and $\mu_1$ for the regular, subregular and identity conjugacy classes. We now write the germ expansion as

$$\Delta^*_G(\gamma)\Phi^{(T,\kappa)}_G(\gamma, f) = \Gamma^{(T,\kappa)}_{reg}(\gamma)\mu_{reg}(f) + \sum_{\pm} \Gamma^{(T,\kappa)}_{sub,\pm}(\gamma)\mu^\pm_{sub}(f).$$

Write $O_{reg}$ for the regular class, and $O_{sub}$ for the union of $O_0$ and $O_1$ in $G$. The derived group of $H$ is isomorphic to $SL(2)$ so that over $\overline{F}$ $H$ has conjugacy classes corresponding to the partitions $2, 1^2$ of 2. These are the regular and subregular classes. The $F$-points of these classes will be denoted $O^H_{reg}$ and $O^H_{sub}$. We also fix invariant forms on these classes and associate invariant measures $\mu^{st}_{reg}(f^H)$, $\mu^{st}_{sub}(f^H)$,

$f^H \in C_c^\infty(H)$. The Shalika germ expansion on $H$ takes the form

$$D_H \Phi^{(T,st)} = \Gamma_{reg}^{(T,st)}(\gamma)\mu_{reg}^{st}(f^H) + \Gamma_{sub}^{(T,st)}(\gamma)\mu_{sub}^{st}(f^H)$$

We will prove theorem 4.2 by showing that with appropriate normalizations of measures

$$\Gamma_{reg}^{(T,\kappa)}(\gamma) = \Gamma_{reg}^{(T,st)}(\gamma)$$

$$\Gamma_{sub,+}^{(T,\kappa)}(\gamma) = \Gamma_{sub}^{(T,st)}(\gamma)$$

$$\Gamma_{sub,-}^{(T,\kappa)}(\gamma) = 0$$

The functions on the left are associated to $G$. Those on the right, although the notation does not indicate it, are germs on $H$.

Once these equalities are demonstrated we select $f \in C_c^\infty(G)$ and select any function $f^H \in C_c^\infty(H)$ such that

$$\mu_{reg}(f) = \mu_{reg}^{st}(f^H)$$

$$\mu_{sub}^{+}(f) = \mu_{sub}^{st}(f^H)$$

Then $D_H \Phi_H^{(T,st)}(\gamma, f^H) = \Delta_G^* \Phi_G^{(T,\kappa)}(\gamma, f)$ wherever the germ expansion is valid.

## 6. A Reduction

Every $f \in C_c^\infty(G)$ may be written as $f = f_1 + f_2, \quad f_i \in C_c^\infty(G)$, where $\mu_{reg}(f_1) = 0$, $\mu_{sub}^{\pm}(f_2) = 0$. This allows us to break the proof of theorem 4.2 into two cases. Section 9 proves theorem 4.2 for functions such that $\mu_{sub}^{+}(f) = \mu_{sub}^{-}(f) = 0$. This section gives a reduction for functions such that $\mu_{reg}(f) = 0$.

**Lemma 6.1.** *Suppose that for each $(T, \kappa)$ associated to $H$ the germs $\Gamma^{(T,\kappa)}_{sub,\pm}(\gamma)$ have the form*

$$\Gamma^{(T,\kappa)}_{sub,\pm}(\exp(\lambda^2 X)) = |\lambda|^2 m^{(T,\kappa)}_{\pm}(X)$$

*where $m^{(T,\kappa)}_{\pm}$ satisfies*

$$m^{(T,\kappa)}_{\pm}(X + Z) = m^{(T,\kappa)}_{\pm}(X)$$

*for $|\lambda| \leq 1$, for $X$, $X + Z$ sufficiently small regular elements of Lie $H'$* and *$Z$ belonging to the Lie algebra of the center of $H'(F) \subseteq G(F)$. Then theorem 4.2 holds for all functions $f$ such that $\mu_{reg}(f) = 0$.*

**Proof.** By [HC] there exists a neighborhood $V$ of the identity 0 in *Lie* $G$ such that for all regular $X \in V \cap Lie\, G$

$$\Gamma^{(T,\kappa)}_{sub,\pm}(\exp(\lambda^2 X)) = |\lambda|^2 \Gamma^{(T,\kappa)}_{sub,\pm}(\exp(X))$$

for $|\lambda| \leq 1$, $\lambda \in F^\times$. Select $z' = \exp(Z) \neq 1$ small enough in $Z(H'(F)) \setminus Z(G(F))$ so that there exists a neighborhood $V_1$ of 0 in *Lie* $G$ satisfying $V_1, Z + V_1 \subseteq V$. Then if $X$ is regular and lies in $V_1 \cap Lie\, H'$

$$\Gamma^{(T,\kappa)}_{sub,\pm}(\exp(X)) = m^{(T,\kappa)}_{\pm}(X) = m^{(T,\kappa)}_{\pm}(X + Z) = \Gamma^{(T,\kappa)}_{sub,\pm}(z' \exp(X))$$

or replacing $X$ by $\lambda X$, $\Gamma_{sub,\pm}^{(T,\kappa)}(\exp(\lambda X)) = \Gamma_{sub,\pm}^{(T,\kappa)}(z'\exp(\lambda X))$. Selecting $f$ as in the hypothesis of the lemma,

$$\Delta_G^*(\exp(\lambda X))\Phi_G^{(T,\kappa)}(\exp(\lambda X), f) = \Delta_G^*(z'\exp(\lambda X))\Phi_G^{(T,\kappa)}(z'\exp(\lambda X), f).$$

Now $C_G(z') = H'$ so by the results of section 4, we see that this integral is equal to a stable orbital integral $D_H(z'\exp(\lambda X))\Phi_H^{(T,st)}(z'\exp(\lambda X), f_1)$ (centered at $z_1$) for some $f_1 \in C_c^\infty(H)$. Replacing $f_1$ by $f^H$ defined by $f^H(x) = f_1(z_1 x)$ and using the fact that $D_H(z'\exp(\lambda X)) = D_H(\exp(\lambda X))$ we see that $f^H \in C_c^\infty(H)$ and that the original $\kappa$-orbital integral is equal to $D_H(\gamma)\Phi_H^{(T,st)}(\gamma, f^H)$.

## 7. Igusa Theory

The idea behind the explicit construction of germs is the following. Fix a curve $\Gamma(F) = \{\exp(\lambda X)\} \subseteq T(F)$. One constructs a variety $Y_1$ whose $F$-points form a compactification of $\Gamma(F) \times (T\backslash G)(F)$. The orbital integral $\Delta_G^*(\gamma)\Phi_G^{(T,\kappa)}(\gamma, f)$ is replaced by an integral over the image of $\{g^{-1}exp(\lambda X)g | g \in (T\backslash G)(F)\}$ in the variety $Y_1$. For $\lambda \neq 0$ this image is the set of $F$-points of a smooth hypersurface $C_\lambda$ of $Y_1$. For $\lambda = 0$, $C_0$ becomes a divisor with normal crossings. As $\lambda$ tends to zero $C_\lambda$ tends to $C_0$ and the integral over $C_\lambda$ tends to an integral over $C_0$. In fact one obtains an expansion for $\Delta_G^*(\gamma)\Phi_G^{(T,\kappa)}(\gamma, f)$ as a sum of integrals taken over the irreducible components of $C_0$ and their intersections. One then observes that this expansion coincides with the Shalika germ expansion, thus obtaining an explicit integral formula for the Shalika germs. From this, the property of section 5 is easily verified.

In the next section we shall give local coordinate charts for a variety $Y_1$ compactifying $\Gamma \times T\backslash G$. It is verified in [L2] that this variety satisfies all of the technical conditions of the theory needed to guarantee that the orbital integral $\Delta_G^*(\gamma)\Phi_G^{(T,\kappa)}(\gamma, f)$ has an expansion in terms of integrals over the irreducible components of $C_0$ and their intersections. We shall take this fact for granted.

To simplify matters we make one minor modification of the theorem of Igusa giving an expansion as set forth in [L2]. There it is required that any irreducible component of $C_0$ with an $F$-rational point be defined over $F$. We shall study the situation in which there are two divisors whose intersection is defined over $F$ which are not themselves defined over $F$.

Let $F$ be a p-adic field of characteristic zero. Let $K/F$ be a degree $r$ cyclic extension with generator $\sigma$ of the Galois group. Let $NK$ denote the norms of elements of $K$ in $F$. Let $\eta_{K/F} : F^\times/NK^\times \to \mathbf{C}^\times$ be a generator of the characters on this quotient. Let $|\ |$ be the absolute value on $F$ normalized by the Haar measure in the usual manner or a compatible extension to $K$. Let

$$U = \{(\mu_1, \dots, \mu_n)| \quad |\mu_i| \leq \epsilon_i, \quad \mu = \mu_1 = \sigma(\mu_2) = \dots = \sigma^{r-1}(\mu_r)\} \subseteq K^r \times F^{n-r}$$

where $\epsilon_1 = \dots = \epsilon_r$. Let $\lambda = \alpha_0 \prod_1^n \mu_i^{a_i}$ where $1 = a_1 = \dots = a_r$, $\alpha_0,\ \lambda \in F^\times$ and assume $|\alpha_0|$ and $\eta_{K/F}(\alpha_0)$ are constant on $U$. Let $\omega = \prod_1^n \mu_i^{b_i} \frac{d\mu_1}{\mu_1} \wedge \dots \wedge \frac{d\mu_n}{\mu_n}$ where $b = b_1 = \dots = b_r$. Suppose that $b_i \neq a_i b$ for $i > r$. Let $f = \kappa_{r+1}(\mu_{r+1}) \cdots \kappa_n(\mu_n)$ for some unitary characters $\kappa_{r+1}, \dots, \kappa_n$ of $F^\times$. Then $f$ is a function on $U$. Write $N\mu = \mu_1 \dots \mu_r$.

**Lemma 7.1.** *In the above setting,* $F(\lambda) =$

$$\int_{U, \lambda = \alpha_0 \mu_1 \cdots \mu_n} f|\frac{\omega}{d\lambda}| = \frac{1}{r} \sum_{j=0}^{r-1} |\lambda|^{b-1} \delta_{K/F} \eta_{K/F}^j(\lambda) F(j) + \sum_{\beta \neq b, \theta, k} \theta(\lambda) |\lambda|^{\beta - 1} m(\lambda)^k F_k(\theta, \beta, f)$$

*for $\lambda$ sufficiently small where*

$$\delta_{K/F} = \int_{N\mu=1} \frac{d\sigma(\mu) d\sigma^2(\mu) \cdots d\sigma^{r-1}(\mu)}{|\sigma(\mu)\sigma^2(\mu) \cdots \sigma^{r-1}(\mu)|}$$

$$F(j) = \int_{U_0} \frac{|\omega_0| f}{\eta_{K/F}^j(\alpha_0 \mu_{r+1}^{a_{r+1}} \cdots \mu_n^{a_n})}$$

$$U_0 = \{(\mu_1, \dots, \mu_n) \in U | \mu_1 = \cdots \mu_r = 0\}$$

$$\omega_0 = \frac{\omega}{\frac{d\mu_1 \cdots d\mu_n}{\mu_1 \cdots \mu_n} |\lambda|^b}$$

*and where $F_r(\theta,\beta,f)$ are appropriate constants.*

**Proof.** Denote the expansion on the right hand side in the lemma by $G(\lambda)$. By the theory of Mellin transforms it is enough to check that

$$F_s = \int_{|\lambda|\le\epsilon} F(\lambda)\theta^{-1}(\lambda)|\lambda|^s d\lambda \quad \text{and} \quad G_s = \int_{|\lambda|\le\epsilon} G(\lambda)\theta^{-1}(\lambda)|\lambda|^s d\lambda$$

have the same principal parts at every complex $s$ for every unitary character $\theta$ of $F^\times$. We shall only treat the terms $|\lambda|^{b-1}\eta^j_{K/F}(\lambda)$ as the form of the other terms is well understood [L2,I]. The terms $|\lambda|^{b-1}\eta^j_{K/F}(\lambda)$ are determined by the principal parts of $F_s$ and $G_s$ at $s=-b$. Thus we shall only consider the principal parts at $s=-b$. Write $x\sim y$ if $x$ and $y$ are meromorphic functions of $s$ having the same principal part at $s=-b$. It is easy to see that if $\theta(N\mu)\ne 1$ then both Mellin transforms are analytic at $s=-b$. So we take $\theta=\eta^j_{K/F}$ for some $j$. We are led to compare the two Mellin transforms:

$$\begin{aligned}
F_s &\sim \int_{|\lambda|\le\epsilon,U} |\lambda|^s\eta^{-j}_{K/F}(\lambda)f|\omega| \\
&= \int_{|\lambda|\le\epsilon,U} |\frac{(N\mu)^{s+b}d\mu_1\dots d\mu_r}{N\mu}| \times \\
&\cdot\int |\alpha_0|^s\eta^{-j}_{K/F}(\alpha_0)\eta^{-ja_{r+1}}_{K/F}\kappa_{r+1}(\mu_{r+1})|\mu_{r+1}|^{a_{r+1}s+b_{r+1}}\frac{d\mu_{r+1}}{|\mu_{r+1}|}\cdots\eta^{-ja_n}_{K/F}\kappa_n(\mu_n)|\mu_n|^{a_ns+b_n}\frac{d\mu_n}{|\mu_n|}
\end{aligned}$$

(which has the same principal part without the constraint $|\lambda|\le\epsilon$) and

$$G_s \sim \frac{1}{r}\int_{|\lambda|\leq\epsilon}|\lambda|^s\eta^{-j}(\lambda)|\lambda|^b\frac{d\lambda}{|\lambda|}\eta^j(\lambda)\delta_{K/F}F(j)=\frac{1}{r}\int_{|\lambda|\leq\epsilon}|\lambda|^{s+b-1}d\lambda\delta_{K/F}F(j).$$

(The other terms of this Mellin trasform are $\sim 0$).

Since $a_{r+i}s+b_{r+i}\neq 0$ at $s=-b$ the terms $\int\eta_{K/F}^{-ja_{r+i}}\kappa_{r+i}(\mu_{r+i})|\mu_{r+i}|^{a_{r+i}s+b_{r+i}}\frac{d\mu_{r+i}}{|\mu_{r+i}|}$ are regular at $s=-b$ and equal the principal-value integral obtained by substituting $-b$ for $s$ ($i>0$). In fact these analytic continuations serve as a definition of these principal-value integrals. The product of these integrals with $|\alpha_0|^s\eta_{K/F}^{-j}(\alpha_0)$ at $s=-b$ is $F(j)=\int_{U_0}\eta^{-j}(\lambda)f|\omega_0|$. Thus $F_s\sim F(j)\int_{|N\mu|\leq\epsilon}|N\mu|^{s+b-1}d\mu_1\dots d\mu_r$. Write $\lambda_0=\mu_1\dots\mu_r$ and substitute for $\mu_1$ to obtain

$$F(j)\int_{|\lambda_0|\leq\epsilon}|N\mu|^{s+b-1}d\mu_1\dots d\mu_n\sim F(j)\delta_{K/F}\int_{\lambda_0\in NK,\ |\lambda_0|\leq\epsilon}|\lambda_0|^{s+b-1}d\lambda_0$$

Now

$$\int_{\lambda\in NK,|\lambda|\leq\epsilon}|\lambda|^{s+b-1}d\lambda=\frac{1}{r}\int_{|\lambda|\leq\epsilon}\sum_0^{r-1}\eta_{K/F}^j(\lambda)|\lambda|^{s+b-1}d\lambda\sim\frac{1}{r}\int_{|\lambda|\leq\epsilon}|\lambda|^{s+b-1}d\lambda.$$

Making this substitution we obtain $F_s\sim G_s$.

## 8. Langlands' variety of stars

Langlands constructs the variety $Y_1$ and gives explicit coordinate patches and coordinates on the variety. We change notation from that used in [L2]. The roots $\alpha$

and $\beta$ of this paper are denoted $\alpha'$ and $\alpha''$ in [L2]. We introduce coordinates $\lambda$, $x(\alpha)$, $x(\beta)$, $x(\gamma)$, $z(\alpha)$, $z(\beta)$, $w(\gamma)$ which are related to the coordinates of [L2] by $x(\alpha) = u$, $x(\beta) = v$, $x(\gamma) = w$, $z(\alpha) = x/b(\lambda)$, $z(\beta) = y/c(\lambda)$, $w(\gamma) = \frac{U}{b(\lambda)c(\lambda)V} = \frac{d(\lambda)-Vc(\lambda)}{b(\lambda)c(\lambda)V}$. Here $b = \frac{1-\alpha^{-1}(s(\lambda))}{\lambda}, c = \frac{1-\beta^{-1}(s(\lambda))}{\lambda}, d = \frac{1-\alpha^{-1}\beta^{-1}(s(\lambda))}{\lambda}$. We set $b(0) = \alpha(X)$, $c(0) = \beta(X)$, $d(0) = \gamma(X)$ so that $\alpha(X) + \beta(X) = \gamma(X)$. These are the positive roots of the Lie algebra evaluated on the tangent direction $X$. In the following sections $\gamma$ always denotes the Lie algebra root $\alpha + \beta$. The coordinate $V$ is defined by $V = \frac{-z_1''}{z_3''}$, where $-\frac{z_1''}{z_3''}$ is a rational function to be defined and discussed in section 8. Our definitions then imply that

$$1 + \frac{z_1''}{z_3''} = \frac{-\alpha(X)}{\beta(X)} \frac{(1 - \beta(X)w(\gamma))}{(1 - \alpha(X)w(\gamma))} \tag{8.1}$$

whenever $\lambda = 0$ and both sides are defined.

With this notation the coordinate relations (4.1)-(4.3) of [L2] become

$$\lambda = z(\alpha)x(\alpha) \tag{8.2}$$

$$\lambda = z(\beta)x(\beta)$$

$$\lambda w(\gamma) = x(\gamma)z(\alpha)z(\beta)$$

Having made this conversion to new notation, the reader is free to forget the previous notation. Only (7.1) and (7.2) need be retained. We will often abbreviate $w(\gamma)$ as $w$.

We note that when $x(\gamma) \neq 0$ , then $z(\alpha) = \frac{x(\beta)w(\gamma)}{x(\gamma)}$, $z(\beta) = \frac{x(\alpha)w(\gamma)}{x(\gamma)}$, and $\lambda = \frac{x(\alpha)x(\beta)w(\gamma)}{x(\gamma)}$. Then $x(\gamma) \neq 0$, $\lambda = 0$ defines three irreducible divisors $E_\alpha$, $E_\beta$, and $E_0$ with normal crossings defined respectively by $x(\alpha) = 0$, $x(\beta) = 0$, $w(\gamma) = 0$. In other words $C_0$ breaks into three irreducible components on this patch. These divisors are called $E_1''$, $E_1'$, and $E_4$ in [L2].

Actually the construction of [L2] leads to nine divisors not just these three. Fortunately, none of the other divisors contribute to the Shalika germ expansion of a $\kappa$-orbital integral. Two of those divisors $E_3$ and $E_5$ have support lying over the identity element. We have shown above that the Shalika germ associated to the identity element is zero. Hence these divisors do not contribute to the germ expansion. Three other divisors $E_2^1$, $E_2^2$, $E_2^3$ have support lying over the subregular unipotent classes. It is shown in [L2] however that their contribution to the asymptotic expansion is by terms of the form $\theta(\lambda)|\lambda|^2$, $\theta$ a character of finite order of $F^\times$. Since there are no such terms in our expansion (see proof of lemma 5), they too make no contribution. Finally there is a divisor $E_6$. It is not actually a divisor on $Y_1$ but on the variety obtained by blowing up along the intersection of $E_\alpha$ and $E_\beta$. Lemma 7.1 allows us to work directly with $Y_1$ -not the blown up variety. Thus the divisor $E_6$ is not present in our construction.

We will confine our remarks to a single patch. The complement of the union of patches isomorphic to this one lies in a union of irreducible divisors excluded in the previous paragraph, and we leave it as an exercise to see that the constructions carried out on this patch extend to the isomorphic patches.

## 9. Regular Classes

This section gives an explicit formula for the function $\kappa(\sigma(g)g^{-1})$ in terms of the transfer factor and the coordinates on $Y_1$. This formula is then used to show that theorem 4.2 holds for functions satisfying $\mu^+_{sub}(f) = \mu^-_{sub}(f) = 0$.

Let $X$ be the tangent direction of the curve $\Gamma$ at the identity. Let $\mathbf{T}$ be the diagonal subgroup of $G$. Let $\pi_2 : H' \to U_E(1)$ be the projection onto the (2,2) matrix coefficient. Let $\eta_{E/F}$ be the nontrivial character on $H^1(F, U_E(1))$ or the nontrivial character of $F^\times$ modulo the norms of $E^\times$ - depending on the context. As in section 2 let

$$\Delta_0(X) = \lim_{\lambda \to 0} \left[ \frac{\Delta(\exp(\lambda X))}{D_{G/H}(\exp(\lambda X))} \right]$$

Let $\Delta_{E_0} = \kappa(\sigma(g)g^{-1})|_{E_0}$. We will see that this limit exists on an open set of $E_0$ and depends on $E_0$ only through the tangent direction $X$. Define $w' =^{def} w/(1+\alpha(X)w)$, $w$ as in section 7.

**Proposition 9.1.**

$$\Delta_0(X)\Delta_{E_0}(X) = 1$$

*and*

$$\begin{aligned} m_\kappa =^{def} \Delta_0(X)\kappa(\sigma(g)g^{-1})|_{\lambda=0} &= \Delta_0(X)\eta_{E/F}(\pi_2(\sigma(g)g^{-1}))|_{\lambda=0} \\ &= \eta_{E/F}((1+\alpha(X)w)(1-\beta(X)w)) \\ &= \eta_{E/F}(\frac{1-\gamma(X)w'}{-w'^2}) \end{aligned}$$

The first part of proposition 9.1 implies the matching of regular germs. The equation states that the transfer factor ($\Delta_0(X)$) times the regular germ of a $\kappa$-orbital integral ($\Delta_{E_0}(X)$) equals the regular germ of a stable orbital integral (1).

The identification of $\Delta_{E_0}$ as the regular germ occurs as follows. The divisor $E_0$ is the only divisor which contributes to the regular germ. The integral over $E_0$ is equal to $\Delta_{E_0}(X)$ times the integral of $f$ over $O_{reg} : \mu_{reg}(f)$.

Similarly one considers a variety $Y_1^H$ which gives the germs of $H$. There is one divisor which contributes to the regular germ. The analysis proceeds as in the case of $G$ except that with stable orbital integrals the term $\kappa(\sigma(g)g^{-1})|_{E_0}$ does not arise. The regular germ for a stable orbital integral is found to be 1 (using the normalizations of measures arising in Igusa theory).

The action of $Gal(\overline{F}/F)$ on $Y_1$ depends on $T$. The action is obtained by twisting an action independent of $T$ by a biregular action of $\Gamma_T \subseteq \Theta = \{1, \omega, \tau, \omega\tau\}$ on $Y_1$. We note here for reference the action of $\Theta$ on the coordinates $w$ and $w'$ at $F$-points of our variety. These relations will be proved in section 10.

| $x$ | $\tau(x)$ | $\tau\omega(x)$ | $\omega(x)$ |
|---|---|---|---|
| $w'$ | $\dfrac{-w'}{1-\gamma(X)w'}$ | $-w'$ | $\dfrac{w'}{1-\gamma(X)w'}$ |
| $\alpha(X)$ | $\beta(X)$ | $-\alpha(X)$ | $-\beta(X)$ |
| $\beta(X)$ | $\alpha(X)$ | $-\beta(X)$ | $-\alpha(X)$ |
| $\gamma(X)$ | $\gamma(X)$ | $-\gamma(X)$ | $-\gamma(X)$ |

| $x$ | $\tau(x)$ | $\tau\omega(x)$ | $\omega(x)$ |
|---|---|---|---|
| $w$ | $-w$ | $-w$ | $w$ |
| $1+\alpha(X)w$ | $1-\beta(X)w$ | $1+\alpha(X)w$ | $1-\beta(X)w$ |
| $1-\beta(X)w$ | $1+\alpha(X)w$ | $1-\beta(X)w$ | $1+\alpha(X)w$ |

We also note that $1+\alpha(X)w = 1/(1-\alpha(X)w')$ and $1-\beta(X)w = \dfrac{1-\gamma(X)w'}{1-\alpha(X)w'}$.

**Proof.** Let **B** be the group of upper triangular matrices. Select $h$ in the derived group of $H'$ such that $T^h = \mathbf{T}$. We may assume that $h \in H'_{der}(K)$ where $K$ is the field splitting $T$ in section 2. From the definition of $\kappa$ in section 1 we see that $\kappa(\sigma(g)g^{-1})$ must equal $\eta_{E/F}(\pi_2(\sigma(g)g^{-1}))$. Writing $g$ in a certain open set of $G$ as $g = htn\nu$ with $t \in T$, $n \in \mathbf{N}$ and $\nu$ in the unipotent radical $\mathbf{N}_\infty$ of the Borel subgroup opposite $\mathbf{B}$ through $\mathbf{T}$ we find that $\sigma(g)g^{-1}$ defines the same cohomology class in $H^1(F,T)$ as $\sigma(h)\sigma(n)\sigma(\nu)\nu^{-1}n^{-1}h^{-1}$. Set $\omega_\sigma = h^{-1}\sigma(h)$. It is an element of the normalizer of $\mathbf{T}$ in $H'_{der}$ depending on $\sigma \in Gal(K/F)$. The image of $\omega_\sigma$ in $\Omega$ coincides with the element in $\{1, \omega\} \subseteq \Omega$ obtained by the map

$$Gal(K/F) \to \Gamma_T \subseteq \Omega \rtimes Gal(E/F) \to \Omega$$

The final map is projection on the first factor. The (2,2) matrix element is not changed by conjugating by $\sigma(h)$; thus $\kappa(\sigma(g)g^{-1}) = \eta_{E/F}(\pi_2(t_\sigma))$ where $t_\sigma = \sigma(n)\sigma(\nu)\nu^{-1}n^{-1}\omega_\sigma \in T^{\sigma(h)} = \mathbf{T}$. By its form, $t_\sigma$ is then uniquely determined by the equation $\mathbf{N}_\infty \mathbf{N} t_\sigma \ni n^{-1}\omega_\sigma$. For those elements $\sigma$ $(\in \{1, \sigma_\tau\} \subseteq Gal(K/F))$ mapping by $\omega_\sigma$ to the trivial element of the Weyl group of $H'$ it follows that $t_\sigma = \omega_\sigma$ lies in the derived group of $H'$ and so $\pi_2(t_\sigma) = 1$. If, on the other hand, $\sigma$ $(\in \{\sigma_\omega, \sigma_{\tau\omega}\} \subseteq Gal(K/F))$ maps to a non-trivial element in the Weyl group of $H'$ then writing

$$n^{-1} = \begin{pmatrix} 1 & -n_\alpha & -n_\gamma + n_\alpha n_\beta \\ 0 & 1 & -n_\beta \\ 0 & 0 & 1 \end{pmatrix}$$

we see that $t_\sigma$ is equal to $diag(x(-n_\gamma + n_\alpha n_\beta), \dfrac{-n_\gamma}{-n_\gamma + n_\alpha n_\beta}, -\dfrac{1}{xn_\gamma})$ where $x$ is a constant depending on $\omega_\sigma$. So $\pi(t_\sigma)^{-1} = (1 - \dfrac{n_\alpha n_\beta}{n_\gamma})$.

Next we relate $\dfrac{n_\alpha n_\beta}{n_\gamma}$ to the coordinate $w(\gamma)$ on the variety. To do so we recall the definition of $\dfrac{z_1''}{z_3''}$. For every element of the Weyl group $\omega$ define $n_\omega \in N_\infty$ by $(\gamma^g, B_+^{\hat{\omega}g}) = (b, \mathbf{B}^{n_\omega})^{\nu'}$ where $\omega = \hat{\omega}^h$, $B_+^h = \mathbf{B}$, and $g = htn\nu$ in a Zariski open set of $G$. Setting $\omega = 1$ we find $\nu = \nu'$. Further $B_+^{\hat{\omega}g} = \mathbf{B}^{\omega n\nu}$ so that $n_\omega n^{-1} \in \mathbf{B}\omega$. Define elements $z_i', z_i''$ by

$$n_\omega = \begin{cases} \exp(z_1' X_{-\alpha}) & \omega = \sigma_\alpha \\ \exp(z_1'' X_{-\beta}) n_{\sigma_\alpha} & \omega = \sigma_\beta\sigma_\alpha \\ \exp(z_2' X_{-\alpha}) n_{\sigma_\beta\sigma_\alpha} & \omega = \sigma_\alpha\sigma_\beta\sigma_\alpha \\ \cdots & \cdots \\ \exp(z_3'' X_{-\beta}) n_{\omega'} & \omega = \sigma_\beta\omega';\ \omega' = \sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta\sigma_\alpha \end{cases}$$

Here

$$\exp(X_{-\alpha}) = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix} \text{ and } \exp(X_{-\beta}) = \begin{pmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix}$$

The condition $n_\omega = 1$ for $\omega = (\sigma_\alpha\sigma_\beta)^3$ by a $3 \times 3$ matrix calculation implies $z_1' + z_2' + z_3' = 0$, $z_1'' + z_2'' + z_3'' = 0$, $z_1' z_3'' = z_2' z_2''$. The condition $n_\omega n^{-1} \in \mathbf{B}_+\omega$ for $\omega = \sigma_\alpha, \sigma_\beta\sigma_\alpha, \sigma_\beta$ implies by a $3 \times 3$ matrix calculation $z_1' = 1/n_\alpha$, $z_1'' = n_\alpha/n_\gamma$,

$z_3'' = -1/n_\beta$ so that by (8.1)

$$1 - \frac{n_\alpha n_\beta}{n_\gamma} = 1 + \frac{z_1''}{z_3''} = \frac{-\alpha(X)}{\beta(X)} \frac{1-\beta(X)w}{1+\alpha(X)w}$$

Thus we see that the cocycle $\pi_2(t_\sigma)$ $(\lambda = 0)$ is given by

$$1, \sigma_\tau \to 1, \qquad \sigma_\omega, \sigma_{\tau\omega} \to \frac{-\beta(X)(1+\alpha(X)w)}{\alpha(X)(1-\beta(X)w)}$$

in $H^1(F, U_E(1))$. Set $t = (\frac{\beta(X)}{1-\beta(X)w}) \in U_E(1, K)$. Then $\pi_2(t_\sigma)\sigma(t)t^{-1}$ is given by

$$1, \sigma_\omega \to 1, \qquad \sigma_\tau, \sigma_{\tau\omega} \to \frac{(1+\alpha(X)w)(1-\beta(X)w)}{\alpha(X)\beta(X)} \in F^\times$$

Thus $\eta_{E/F}(\pi_2(t_\sigma))$ is equal to

$$\eta_{E/F}(\frac{(1+\alpha(X)w)(1-\beta(X)w)}{\alpha(X)\beta(X)})$$

Inserting the norm $-w^2$ we find the equivalent expressions

$$\eta_{E/F}(\frac{(1+\alpha(X)w)(1-\beta(X)w)}{\alpha(X)\beta(X)(-w^2)}) = \eta_{E/F}(\frac{1-\gamma(X)w}{-\alpha(X)\beta(X)w'^2})$$

The divisor $E_0$ is defined by $w = 0$. Thus

$$\Delta_{E_0}(X) = \eta_{E/F}(\frac{(1+\alpha(X)w)(1-\beta(X)w)}{\alpha(X)\beta(X)})|_{w=0} = \eta_{E/F}(\frac{1}{\alpha(X)\beta(X)})$$

Also $\Delta_0(X) = \eta_{E/F}(\alpha(X)\beta(X))$ so that $\Delta_{E_0}(X)\Delta_0(X) = 1$. This completes the proof.

## 10. Actions of the Galois Group

We reproduce here for reference the conditions in [L2] on the coordinates for a point in $Y_1$ to be in $Y_1(F)$. They are determined by the requirement that the embedding $\Gamma_0 \times T\backslash G \to Y_1$ be defined over $F$. Once the action on coordinates is determined, the condition for rationality will be $\sigma_\rho(x) = \rho(x)$ for $\rho \in \Gamma_T$ and $x$ a coordinate of $Y_1$. To calculate these conditions for rationality it is enough to do so for the generators $\tau$ and $\omega$.

Begin with $\sigma_\tau$, $\Gamma_T = \{1, \sigma_t\}$. The condition for rationality of $(b, B^{n_\omega})^\nu$ is determined by $b = J^t\overline{(b)}^{-1}J$ and $n_\omega\nu = J^t\overline{(n_\omega\nu)}^{-1}J$, where $(\bar{x})$ denotes conjugation coefficient-wise. In other words $\sigma_\tau$ twists the ordinary conjugation by the algebraic action $\tau :\ b \to J^t b^{-1} J$, $\tau : n_\omega\nu \to J^t(n_\omega\nu)^{-1}J$. Writing $b = t\begin{pmatrix} 1 & x(\alpha) & x(\gamma) \\ & 1 & x(\beta) \\ & & 1 \end{pmatrix}$, we see that $\tau(x(\alpha)) = -x(\beta)$, $\tau(x(\gamma)) = -x(\gamma)(1 - \frac{x(\alpha)x(\beta)}{x(\gamma)})$. On $E_\alpha \cup E_\beta$, $x(\alpha)x(\beta) = 0$ so that on this set $\tau(x(\gamma)) = -x(\gamma)$. From (8.2) it follows that on $E_\alpha \cup E_\beta$ that $\tau(w(\gamma)) = \tau(\frac{\lambda x(\gamma)}{x(\alpha)x(\beta)}) = \frac{-\lambda x(\gamma)}{x(\alpha)x(\beta)} = -w(\gamma)$. This action of $\tau$ is independent of $T$.

When $\sigma = \sigma_\omega$ then $\omega = \sigma_\alpha\sigma_\beta\sigma_\alpha$ where $\sigma_\alpha$ (resp. $\sigma_\beta$) is the simple reflection associated to the root $\alpha$ (resp. $\beta$). The action of $\omega$ factors by $\sigma_\alpha\sigma_\beta\sigma_\alpha$. The action of $\sigma_\alpha$ on $x(\alpha)$, $x(\beta)$, $x(\gamma)$ is determined, according to [L2], by the condition

$$\begin{pmatrix} 1 & & \\ z & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1 & t_1 x(\alpha) & t_3 x(\gamma) \\ & t_2 & t_2 x(\beta) \\ & & t_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ -z & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} t_1' & & \\ & t_2' & \\ & & t_3' \end{pmatrix} \begin{pmatrix} 1 & x(\alpha)' & x(\gamma)' \\ & 1 & x(\beta)' \\ & & 1 \end{pmatrix}$$

Here $x(\alpha)' = \sigma_\alpha(x(\alpha))$, etc. By equating coefficients of these matrices one finds first that $z = \dfrac{\alpha(X)\lambda}{x(\alpha)}$ and then that $\dfrac{x(\alpha)'}{x(\alpha)} = 1$, $\dfrac{x(\beta)'}{x(\beta)} = (1+\alpha(X)w)$, $\dfrac{x(\gamma)'}{x(\gamma)} = 1$ when $\lambda = 0$. So $\sigma_\alpha(w) = \sigma_\alpha(\dfrac{\lambda x(\gamma)}{x(\alpha)x(\beta)}) = \dfrac{\lambda x(\gamma)}{x(\alpha)x(\beta)(1+\alpha(X)w)} = \dfrac{w}{1+\alpha(X)w}$ when $\lambda = 0$.

Similarly the calculation of the action of $\sigma_\beta$ on $x(\alpha)$, $x(\beta)$, $x(\gamma)$ is identical except that $\begin{pmatrix} 1 & & \\ z & 1 & \\ & & 1 \end{pmatrix}$ is replaced by $\begin{pmatrix} 1 & & \\ & 1 & \\ & z' & 1 \end{pmatrix}$. Equating coefficients of the $3 \times 3$ matrices one obtains on $\lambda = 0$: $z' = \dfrac{\lambda\beta(X)}{x(\beta)}$, $\dfrac{x(\alpha)'}{x(\alpha)} = 1 + \alpha(X)w$, $\dfrac{x(\beta)'}{x(\beta)} = 1$, $\dfrac{x(\gamma)'}{x(\gamma)} = 1$. Then similarly, $\sigma_\beta(w) = \dfrac{w}{1-\beta(X)w}$. Finally, $\sigma_\alpha\sigma_\beta\sigma_\alpha(w) = \sigma_\alpha\sigma_\beta(\dfrac{w}{1+\alpha(X)w}) = \sigma_\alpha(\dfrac{w}{1+\alpha(X)w}) = w$. This leads immediately to the table of section 9.

## 11. Subregular Classes

By the action of the Galois group of the splitting field of $T$ calculated in section

10

$$\sigma_\omega[(1-\beta(X)w)x(\alpha)] = (1-\beta(X)w)x(\alpha) \in E$$

and

$$\sigma_\tau[(1-\beta(X)w)x(\alpha)] = -(1-\alpha(X)w)x(\beta).$$

So

$$\lambda = \frac{x(\alpha)x(\beta)w(\gamma)}{x(\gamma)} = -\frac{[(1-\beta(X)w)x(\alpha)]\sigma_\tau[(1-\beta(X)w)x(\alpha)]w}{(1-\beta(X)w)(1+\alpha(X)w)x(\gamma)}.$$

Consequently

$$\eta_{E/F}(\lambda) = \eta_{E/F}\left(\frac{-w}{(1-\beta(X)w)(1+\alpha(X)w)x(\gamma)}\right).$$

By the theory of Igusa outlined in section 7 (with $K = E, r = 2$) we obtain the formula

$$\sum_{\pm}\Gamma^{(T,\kappa)}_{sub,\pm}(\gamma)\mu^{\pm}_{sub}(f) =$$

$$\frac{\delta_{E/F}}{2}|\lambda|\int_{E_\alpha\cap E_\beta} m_\kappa f|\omega_0| + \frac{\delta_{E/F}}{2}|\lambda|\eta_{E/F}(\lambda)\int_{E_\alpha\cap E_\beta} m_\kappa \frac{f|\omega_0|}{\eta_{E/F}(\lambda)}. \qquad (*)$$

Now $|\omega_0| = \frac{d\omega}{|w|^2}\mu^{+}_{sub}$, and $m_\kappa$ is the factor of proposition 9.1. The first term of $(*)$ is

$$\frac{\delta_{E/F}}{2}|\lambda|\int_{\mathbf{P}^1=E_\alpha\cap E_\beta(u)} \eta_{E/F}\left(\frac{1-\gamma(X)w'}{-w'^2}\right)\frac{dw'}{|w'|^2}\ \mu^{+}_{sub}(f).$$

The second term is

$$\frac{\delta_{E/F}}{2}|\lambda|\eta_{E/F}(\lambda)\mu^{-}_{sub}(f)\int_{\mathbf{P}^1}\eta_{E/F}\left(\frac{(1+\alpha(X)w)(1-\beta(X)w)}{\lambda x(\gamma)}\right)\frac{dw}{|w|^2}$$

$$= \frac{\delta_{E/F}}{2}|\lambda|\eta_{E/F}(\lambda)\mu^{-}_{sub}(f)\int_{\mathbf{P}^1}\frac{\eta_{E/F}(-w)dw}{|w|^2} = 0.$$

Note that $\eta_{E/F}(x(\gamma))$ has been pulled in front of the integral over $w$. The factor $\eta_{E/F}(x(\gamma))$ is constant on each $F$-class and gives rise to the signed measure $\mu^{-}_{sub}$. This integral vanishes because for any non-trivial quasi-character $\theta : F^\times \to \mathbf{C}^\times$:

$$\int_{\mathbf{P}^1}\frac{\theta(w)dw}{|w|} =^{(w\to\alpha w)} \int_{\mathbf{P}^1}\theta(\alpha w)\frac{d(\alpha w)}{|\alpha w|}$$
$$= \theta(\alpha)\int_{\mathbf{P}^1}\theta(w)\frac{dw}{|w|}.$$

So that $\theta(\alpha) \neq 1$ implies $\int_{\mathbf{P}^1}\theta(w)\frac{dw}{|w|} = 0$. A more rigorous treatment of this integral is given in [LS1]. Thus we have that the subregular germ is

$$\Gamma^{(T,\kappa)}_{sub,+}(\gamma)\mu^{+}_{sub}(f) = \frac{\delta_{E/F}}{2}|\lambda|\int_{\mathbf{P}^1}\eta_{E/F}\left(\frac{1-\gamma(X)w'}{-w'^2}\right)\frac{dw'}{|w'|^2}\mu^{+}_{sub}(f).$$

This clearly depends only on $X$ through $\gamma(X)$. By lemma 6.1, this completes the proofs of theorems 4.2 and 1.1.

To check the constants explicitly that are involved in the transfer consider the anisotropic Cartan subgroup $T$ split by a quadratic extension $E$ with Galois group $\{1, \sigma_{\omega\tau}\}$. Then

$$\gamma(X), w' \in E, \quad \sigma_{\omega\tau}(\gamma(X)) = -\gamma(X), \quad \sigma_{\omega\tau}(w') = -w'.$$

Then $-w'^2 \in NE$ and $u = \gamma(X)w'$ is a variable over $F$ so that

$$\Gamma_+ = \frac{|\lambda|}{2}\delta_{E/F}\int \eta_{E/F}\left(\frac{1-\gamma(X)w'}{-w'^2}\right)\frac{dw'}{|w'|^2} =$$

$$\frac{|\lambda\gamma(X)|}{2}\int_{\sigma_{\omega\tau}(x)x=1}\frac{dx}{|x|}\int_{\mathbf{P}^1}\eta_{E/F}(1-u)\frac{du}{|u|^2}.$$

It is also shown in [LS1] that the integral $\int_{\mathbf{P}^1}\frac{du}{|u|^2}$ vanishes. Also $\frac{\eta_{E/F}(1-u)+1}{2} = 0$ if and only if $1-u = \sigma_{\omega\tau}(a_0)a_0 \in NE$ we may write (setting $xa_0 = a$)

$$\Gamma_+/|\gamma(X)\lambda| = \int_{\sigma_{\omega\tau}(x)x=1}\frac{dx}{|x|}\int\frac{\eta_{E/F}(1-u)+1}{2}\frac{du}{|u|^2} = \int_{\sigma_{\omega\tau}(a)a=1-u}\frac{da}{|a|}\int_{\mathbf{P}^1}\frac{du}{|u|^2}.$$

Set $b = a/(1-u) \in E$ (so $u = 1-a/b$).

$$\Gamma_+/|\gamma(X)\lambda| = \int_{\sigma_{\omega\tau}(a)=b^{-1}}\frac{da}{|a|}\int\frac{d(a/b)}{|1-a/b|^2} = \int_{Q(F)}\frac{da\,db}{|a-b|^2}$$

where $Q(F)$ is the form of $\mathbf{P}^1\times\mathbf{P}^1$ defined by the rationality condition $\sigma_{\omega\tau}(a,b) = (b^{-1},a^{-1})$ on inhomogenious coordinates.

The integral $\int_{Q(F)}\frac{dadb}{|a-b|^2}$ arises in the paper [LS1] in connection with the germs of $SL(2)$. In that paper it is proved, using Igusa theory, that

$$|\gamma(X)\lambda|\int_{Q(F)}\frac{dadb}{|a-b|^2}$$

is equal to the germ in $SL(2)$ associated to the subregular conjugacy class and the torus split by $E$. Now $\gamma(X)$ denotes the simple positive root of $SL(2)$. The derived group of the quasi-split group $H'$ is isomorphic to $SL(2)$ so that this is also equal to the germ associated to the subregular class on $H'$. (We have used the result, proved by Harish-Chandra [HC], that germs are independent of the center in a suitable sense). Set

$$u_\epsilon(x,z) = \begin{pmatrix} 1 & x & \epsilon z - \frac{x\bar{x}}{2} \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix} \in U_E(3,F)$$

where $x \in E$, $z \in F$, $\bar{\epsilon} = -\epsilon$. Also set $\ell_\epsilon(x,z) = {}^t u_\epsilon(x,z)$. By comparing the normalization of measures used in [LS1] with the normalization of measures used here, we obtain the following explicit form of the matching of orbital integrals near the identity:

**Theorem 11.1.** *Suppose that $f, f^H$ are chosen so that*

$$\mu_{reg}(f) = \mu^H_{reg}(f), \quad \mu^+_{sub}(f) = \mu^H_{sub}(f^H) \tag{1}$$

$$\lim_{\gamma\to 1} \Delta^*_G(\gamma)\Phi^{(T,\kappa)}_G(\gamma, f) = \mu_{reg}(f) \tag{2}$$

$$\lim_{\gamma\to 1} D_H(\gamma)\Phi^{(T,st)}_H(\gamma, f^H) = \mu^H_{reg}(f^H). \tag{3}$$

Suppose also that measures are normalized by the invariant forms

$$\mu_{reg}: \; dzdxdsdt\text{coordinates}: \quad \ell_\epsilon(t,s)u_\epsilon(x,z)\ell_\epsilon(t,s)^{-1}$$

$$\mu_{sub}^{+}: \; zdzdtds\text{coordinates}: \quad \ell_\epsilon(t,s)u_\epsilon(0,z)\ell_\epsilon(t,s)^{-1}$$

$$\mu_{reg}^{H}: dudx\text{coordinates}: \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}$$

$$\mu_{sub}^{H}(f) = f(1).$$

Then

$$\Delta_G^*(\gamma)\Phi_G^{(T,\kappa)}(\gamma, f) = D_H(\gamma)\Phi_H^{(T,st)}(\gamma, f^H)$$

in a sufficiently small neighborhood of the identity.

## 12. Normalization of Measures and the Hecke algebra

The results of this section assume that $E/F$ is an unramified extension. If $M$ is a reductive group over the integers $O_F$ of $F$ set

$$[M] = \frac{\#M(\mathbf{F}_q)}{q^{dim\ M}}.$$

For instance

$$[GL(n)] = (1 - \frac{1}{q^n})(1 - \frac{1}{q^{n-1}})\cdots(1 - \frac{1}{q})$$

and

$$[U_E(3)] = (1 + \frac{1}{q^3})(1 - \frac{1}{q^2})(1 + \frac{1}{q})$$

Here $q$ is the cardinality of the residue field $\mathbf{F}_q$ of $O_F$.

Let $1_G$ denote the normalized characteristic function $1_G = \frac{char\ K}{[G]}$, $K = U_E(3,F) \cap GL(3,O_F)$. Similarly put $1_H = \frac{char K'}{[H]}$, $K' = H' \cap GL(3,O_F)$. Normalize $\omega_{T\backslash G}$ and $\omega_{T\backslash H}$ by

$$\lim_{\gamma \to 1} \Delta_G^*(\gamma)\Phi_G^{(T,\kappa)}(\gamma, f) = \mu_{reg}(f)$$

$$\lim_{\gamma \to 1} \Delta_H^*(\gamma)\Phi_H^{(T,st)}(\gamma, f^H) = \mu_{reg}^{st}(f^H)$$

This section proves

**Theorem 12.1.**

$$\Delta_G^* \Phi_G^{(T,\kappa)}(\gamma, 1_G) = D_H \Phi_H^{(T,st)}(\gamma, 1_H)$$

*for $\gamma$ sufficiently small and $G$-regular.*

This is a weak form of the fundamental lemma. In the case of $SL(n)$ the assumption that $\gamma$ is close to the identity can be removed and consequently for $SL(n)$ the matching of germs implies the fundamental lemma for the identity of the Hecke algebra [H3].

This theorem gives a precise statement of how the measures arising in Igusa theory relate to the maximal compact subgroup. The theorem is actually a statement about germs and is proved by computing the integrals $\mu_{reg}(f)$, $\mu_{sub}^{\pm}, \ldots$.

As a corollary we obtain the alternate form of theorem 12 which holds true for any normalization of measures $\mu_{reg}, \mu_{sub}, \ldots$.

**Corollary 12.2.** *Suppose that*

$$\frac{\mu_{reg}(f)}{\mu_{reg}(1_G)} = \frac{\mu^{st}_{reg}(f^H)}{\mu^{st}_{reg}(1_H)}$$

$$\frac{\mu^{+}_{sub}(f)}{\mu^{+}_{sub}(1_G)} = \frac{\mu^{st}_{sub}(f^H)}{\mu^{st}_{sub}(1_H)}$$

*Also suppose that the measures of the orbital integrals are normalized so as to satisfy:*

$$\lim_{\gamma\to 1} \frac{\Delta^*_G \Phi^{(T,\kappa)}_G(\gamma, 1_G)}{\mu_{reg}(1_G)} = \lim_{\gamma\to 1} \frac{D_H \Phi^{(T,st)}_H(\gamma, 1_H)}{\mu^{st}_{reg}(1_H)} = 1$$

*Then* $\Delta^*_G \Phi^{(T,\kappa)}_G(\gamma, f) = D_H \Phi^{(T,st)}_H(\gamma, f^H)$ *in a sufficiently small neighborhood of* $1$.

**Proof.** (of corollary). Take germ expansions of both sides in theorem 12.1.

**Proof.** (of theorem). By Rao [Ra] for $f \in C^\infty_c(G)$, $f^H \in C^\infty_c(H')$,

$$f \to \int_{K,E,F} f(k^{-1}u_\epsilon(x,z)k)dkdxdz = \mu^{Rao}_{reg}(f)$$

$$f \to \int_{K,F} f(k^{-1}u_\epsilon(0,z)k)dkdz = \mu^{Rao}_{sub}(f)$$

$$f^H \to \int_{K',F} f^H(k^{-1}u_\epsilon(0,z)k)dkdz = \mu^{Rao,st}_{reg}(f^H)$$

$$f^H \to \int_{K'} f^H(1)dk = \mu^{Rao,st}_{sub}(f^H)$$

are invariant measures on $O_{reg}$ , $O_{sub}$, $O^H_{reg}$, $O^H_{sub}$ respectively and so are equal to $\mu_{reg}$, $\mu^+_{sub}$, $\mu^{st}_{reg}$, $\mu^{st}_{sub}$ resp. up to scalars. Let $\chi$ be the characteristic function of

$$\{\ell_\epsilon(t,s)u_\epsilon(x,z)\ell_\epsilon(t,s)^{-1}|t,x\in O_E;s,z\in O_F\}$$

on $G$, and let $\chi'$ be the characteristic function of

$$\{\ell_\epsilon(0,s)u_\epsilon(0,z)\ell_\epsilon(0,s)^{-1}|z,s\in F\}$$

on $H'$. If $\chi(k^{-1}u_\epsilon(x,z)k)\neq 0$ with $x\neq 0$ then $x\in O_E$, $y\in O_F$ and $k=k_bk_\ell$, $k_b\in K\cap \mathbf{B}$, $k_\ell\in K\cap\mathbf{N}_\infty$. Similarly if $\chi'(k'^{-1}u_\epsilon(0,z)k')\neq 0$ with $z\neq 0$ then $z\in O_F$ and $k'=k'_bk'_\ell$, with $k_b\in K'\cap\mathbf{B}$, $k_\ell\in K'\cap\mathbf{N}_\infty$. We use normalizations of measures that assign volume one to the set of elements in $K, K'$ that are congruent to the identity modulo a uniformizing element.

Then

$$\begin{aligned}\mu_{reg}(1_G) &= \frac{\mu_{reg}(\chi)}{\mu^{Rao}_{reg}(\chi)}\mu^{Rao}_{reg}(1_G)\\ &= \frac{\int_{O_F,O_F,O_E,O_E} dsdzdxdt}{\int_{K\cap\mathbf{B}\cdot K\cap\mathbf{N}_\infty} dk\int_{O_F,O_E} dzdx}\frac{\int_K dk\int_{O_F,O_E} dzdx}{[G]}\\ &= \frac{1}{(q+1)(q^2-1)q^6}\frac{[G]q^{dim\ G}}{[G]} = \frac{1}{(1+\frac{1}{q})^2(1-\frac{1}{q})} = \frac{1}{[\mathbf{T}]}\end{aligned}$$

$$\begin{aligned}\mu_{sub}(1_G) &= \frac{\mu_{sub}(\chi)}{\mu^{Rao}_{sub}(\chi)}\mu^{Rao}_{sub}(1_G)\\ &= \frac{\int_{O_F} zdz}{\int_{K\cap\mathbf{B}\cdot K\cap\mathbf{N}_\infty} dk\int_{O_F} zdz}\frac{\int_K dk\int zdz}{[G]} = \frac{1}{(1+\frac{1}{q})}\mu_{reg}(1_G)\end{aligned}$$

$$\mu_{reg}^{st}(1_H) = \frac{\mu_{reg}(\chi')}{\mu_{reg}^{Rao}(\chi')}\mu_{reg}^{Rao,st}(1_H)$$

$$= \frac{\int_{O_F,O_F} dzds}{\int_{K'\cap\mathbf{B}\cdot K'\cap\mathbf{N}_\infty} dk \int_{O_F} dz} \frac{\int_{K'} dk \int_{O_F} dz}{[H]}$$

$$= \frac{1}{(q+1)(q^2-1)q^2}\frac{[H]q^{dim\ H}}{[H]} = \frac{1}{(1+\frac{1}{q})^2(1-\frac{1}{q})} = \frac{1}{[\mathbf{T}]}$$

$$\mu_{sub}^{st}(1_H) = \frac{\mu_{sub}^{st}(\chi')}{\mu_{sub}^{Rao}(\chi')}\mu_{sub}^{Rao,st}(1_H)$$

$$= \frac{1}{\int_{K'} dk}\frac{[H]q^{dim\ H}}{[H]}$$

$$= \frac{q^{dim\ H}}{(q^2-1)(q^2+q)(q+1)} = \frac{1}{(1+\frac{1}{q})}\mu_{reg}^{st}(1_H)$$

So

$$\Delta_G^*\Phi_G^{(T,\kappa)}(\gamma, 1_G) = \frac{1}{[\mathbf{T}]} + \frac{1}{[H]}\Gamma_+ = D_H\Phi_H^{(T,st)}(\gamma, 1_H)$$

for $\gamma$ small.

## 13. Other endoscopic groups

The endoscopic groups $H$ of $U_E(3)$ are found among the endoscopic groups $H$ of $SU_E(3)$ (allowing for minor modifications of the centers of $H$).

Upon extension of scalars to $E$ the endoscopic groups of $SU_E(3)$ become endoscopic groups of $SL(3)/E$. It is known that these are either a split Levi factor of $SL(3)$ or a torus split by a cubic cyclic extension. Descending back down to $SU_E(3)$ it is not difficult to check that the endoscopic groups must be one of the following:

1. $SU_E(3)$ itself. (This group is quasi-split and has no inner forms).
2. the subgroup $\left\{\begin{pmatrix} * & & * \\ & * & \\ * & & * \end{pmatrix}\right\}$
3. a Cartan subgroup of $SU(3)$ contained in a Borel subgroup over $F$
4. a Cartan subgroup of $SU(3)$ split by an $S_3$ extension of $F$ which becomes cyclic over $E$.

Cases (1) and (3) correspond to Levi factors of $SL(3)$ over $E$. The transfer of (3) is easily dealt with using arguments of [HC].

Case (2), at least for $G = U_E(3)$ has been dealt with fully above. The argument for $G = SU_E(3)$ is identical, and is easily deduced from the above.

Case (4). Note that $U_E(3)$ is isomorphic to $GL(3)$ over $E$. Consequently this class of endoscopic groups does not arise for $U_E(3)$ (every endoscopic group of $GL(3)$ is a split Levi factor). Similarly $G_{adj}$ does not have any endoscopic Cartan subgroups of this type.

Proving the transfer to a Cartan subgroup is equivalent to showing that the germs $\Gamma_O^{(T,\kappa)}(\gamma) = 0$ vanish when $O$ is not regular. For then the orbital integral becomes a locally constant function of $\gamma \in T(F)$.

In our situation $G = SU_E(3)$ there is a simple symmetry which forces $\Gamma_{sub,\pm}^{(T,\kappa)} =$

0 and which gives the result.

**Lemma 13.1.** *If $h \in G(\overline{F})$ and $\sigma(h)h^{-1} \in H^1(F, Z_G)$ then*

$$\Gamma_{O^h}^{(T,\kappa)} = \Gamma_O^{(T,\kappa)} \kappa(\sigma(h)h^{-1})$$

**Proof.** Compare orbital integrals of $f$ and $f_h$ where $f_h(x) = f(x^h)$. See [H2].

For $g \in G_{adj}(\overline{F})$ let $\hat{g} \in G(\overline{F})$ denote a lift. Let $\hat{G}_{adj}(F)$ denote the inverse image of $G_{adj}(F)$ in $G(\overline{F})$. Then $\hat{G}_{adj}(F) = \{h \in G(\overline{F}) | \sigma(h)h^{-1} \in Z_G\}$. If $O$ is subregular then $O^h = O$ for $h \in \hat{G}_{adj}(F)$ as can be seen by conjugating $\begin{pmatrix} 1 & 0 & ux \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ by $diag(a, b, c) \in \hat{G}_{adj}(F)$ as in section 4. Thus we are reduced to finding $h \in \hat{G}_{adj}(F)$ with $\kappa(\sigma(h)h^{-1}) \neq 1$.

There is an isomorphism over $F$: $(T \backslash G)(F) \simeq (T_{adj} \backslash G_{adj})(F)$. Then there is a well defined function $m_{\kappa'}$ on $(T_{adj} \backslash G_{adj})(F)$ given by $m_{\kappa'}(T_{adj} g) = \kappa(\sigma(\hat{g})\hat{g}^{-1})$. If $\kappa(\sigma(h)h^{-1}) = 1$ for all $h \in \hat{G}_{adj}(F)$ then $m_{\kappa'}$ factors through $\kappa' : Z^1(T_{adj}) \to \mathbf{C}$, $m_{\kappa'}(T_{adj} g) = \kappa(\sigma(\hat{g})\hat{g}^{-1})$. $\kappa'$ is trivial on coboundaries so descends to $H^1(F, T_{adj})$. But then $(\kappa_{adj}, T_{adj})$ defines an endoscopic group of $G_{adj}$ falling into case (4) above, contradicting the remarks above.

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